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Correlation functions in holographic RG flows

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Abstract: We discuss the computation of correlation functions in holographic RG flows. The method utilizes a recently developed hamiltonian version of holographic renormalization and it is more efficient than previous methods. A significant simplification concerns the treatment of infinities: instead of performing a general analysis of counterterms, we develop a method where only the contribution of counterterms to any given correlators needs to be computed. For instance, the computation of renormalized 2-point functions requires only an analysis at the linearized level. We illustrate the method by discussing flat and AdS-sliced domain walls. In particular, we discuss correlation functions of the Janus solution, a recently discovered non-supersymmetric but stable AdS-sliced domain wall.

Keywords: AdS-CFT and dS-CFT Correspondence, Supergravity Models.

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Contents

1. Introduction

One of the successes of the gravity/gauge theory correspondence is the ability to perform detailed computations of QFT correlation functions by doing a classical gravitational computation. The holographic prescription put forward in [1, 2] identifies the boundary values of bulk fields with sources of gauge invariant operators of the boundary theory and the bulk partition function (which is a functional of the boundary data) with the generating functional of correlation functions of gauge invariant operators. In particular, the supergravity on-shell action is identified with the generating functional of connected graphs of the (strongly coupled) boundary QFT at large-N.

The correlation functions computed using this prescription exhibit divergences due to the infinite volume of spacetime and need to be renormalized. A systematic method for doing this was developed in [3, 4, 5] following earlier work in [6], see [7] for a review. The method is complete and can be used to compute renormalized correlation functions in any holographic RG-flow described by a (locally) asymptotically AdS (AAdS) spacetime. Some of the steps, however, are rather tedious which make the method difficult to apply. In this paper we will present a significantly simpler version of the method.

Let us first recall the computation of 2-point functions as developed in the early period of the AdS/CFT correspondence [1, 8]. We will call this method the "old approach". To regulate the theory one imposes a cutoff at large radius and solves the linearized fluctuation equation with Dirichlet boundary condition at the cutoff. The second variation of the cutoff on-shell action is then computed in momentum space yielding an expression containing singular powers of the cutoff times integer powers of p plus non-singular terms in the cutoff which are non-analytic in p. The 2-point function is defined as the leading non-analytic term. The polynomial terms in p which are dropped are contact δ-function terms in the position space correlator, which are scheme dependent in field theory and largely unphysical. The non-analytic term has an absorptive part in p which correctly gives the 2-point function for separated points in x-space. This method is quite efficient, but it is not fully satisfactory since it is not correct in general to simply drop divergent terms in correlation functions. The subtractions should be consistent with each other and they should respect all (non-anomalous) symmetries.

In holographic renormalization one replaces the above computation by a two step procedure. In the first step renormalization is performed. This is done via the so-called near-boundary analysis. This step associates to a given bulk action a set of universal local boundary counterterms that render the on-shell action finite on an arbitrary solution of the bulk equations of motion. Furthermore, to each bulk field we associate a pair of conjugate variables: the "source" and the exact 1-point function in the presence of sources (or "vev"). These can be read off from the asymptotic expansion of the bulk fields.

n-point functions can now be obtained by functionally differentiating the 1-point function w.r.t. the sources. The near-boundary analysis leaves undetermined the vev part of the solution. To obtain the vev part in terms of the source part we need an exact (as opposed to asymptotic) solution of the bulk field equations with arbitrary Dirichlet data. Such computation is (presently) not possible in all generality but one can solve the bulk equations perturbatively around a given solution. In particular, to obtain 2-point functions it is sufficient to obtain a solution of the linearized field equations, thus connecting with the method described earlier. Since all subtractions are made by means of covariant counterterms mutual consistency is guaranteed. Extra bonus of the method is that all Ward identities and anomalies are manifest ab initio and 1-point functions (with sources equal to zero) are automatically computed. Furthermore, the procedure demonstrates that holography exhibits general field theoretic features such as the fact that the cancellation of UV divergences does not depend on the IR physics. The main drawback of the method is that the near-boundary analysis is somewhat tedious to carry out.

It is conceptually satisfying that the procedure of renormalization can be carried out in full generality without reference to a particular solution, i.e. that the counterterms are associated to a bulk action, not a particular solution. On the other hand, however,

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interesting solutions corresponding to different RG-flows usually involve different sets of fields and different actions. To compute correlation functions in different RG-flows one would thus need to first complete the near-boundary analysis for each different action. Furthermore, some counterterms may have vanishing contribution when we consider a specific solution and may contribute to only a few correlation functions. Thus, if we are only interested in the computation of a small set of correlation functions it would be more efficient if we had a method that computes only the contribution of the counterterms to these correlators (rather than first computing all counterterms and then specializing to the specific case). The main result of this work is the development of such a method.

In [9], extending previous work [10, 11, 12] (see also [13]), we developed a hamiltonian method to obtain renormalized correlation functions of the dual quantum field theory from classical AdS gravity. The method is based on a hamiltonian treatment of gravity using the AdS radial coordinate as the 'time' coordinate. More concretely, the canonical momenta conjugate to the bulk fields evaluated on a regulating hypersurface are identified with the (regulated) one point functions. These are functionals of the bulk fields on the same hypersurface and the bulk field equations become first order functional differential equations for the momenta. To obtain covariant counterterms one expresses the radial derivative in terms of functional derivatives w.r.t. the bulk fields. The Dirichlet boundary conditions on the bulk fields then imply that the radial derivative is asymptotically identified with the total dilatation operator of the dual field theory. This is indeed consistent with the interpretation of the bulk radial coordinate as the energy scale of the dual field theory. The momenta are then expanded in covariant eigenfunctions of the dilatation operator, which is the analogue of the asymptotic expansion of the bulk fields in the standard approach. However, since the expansion is by construction covariant, all counterterms, including those related to the conformal anomaly, are directly computed without the need to invert the asymptotic expansions. The method, as the standard method of holographic renormalization, makes all (anomalous and non-anomalous) Ward identities manifest and in fact we showed that the two approaches are completely equivalent. Nevertheless, the new method leads to a much more efficient algorithm for the computation of correlation functions from holography as we will demonstrate in this paper by various examples. The main advantage is that the divergences of the holographic correlators can be removed consistently without the need for a general near boundary analysis as was done in previous methods.

The paper is organized as follows. In section 1 we recall some of the highlights of the method developed in [9] to the extent that is necessary for the subsequent discussion. In section 2, we consider Poincaré domain wall solutions of the bulk supergravity theory with a single active scalar field turned on. We discuss how to distinguish between explicit breaking of the conformal symmetry by an operator deformation and spontaneous breaking by vevs by computing the exact one-point functions for these backgrounds. Two point functions are then calculated and we show explicitly how they can be renormalized without the need for a general near boundary analysis. Finally, in section 4 we study the recently found Janus solution [14], which is a particular non-supersymmetric but stable AdS domain wall [15]. We calculate the vevs of the background as well as some two-point functions and we show that the Ward identities associated with the symmetries of the background are satisfied.

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2. Hamiltonian holographic renormalization

We start by considering the truncated supergravity action describing gravity in (d+1)-dimensions coupled to a single scalar field:

$$S[g,\Phi]=\int_M d^{d+1}x\sqrt{g}\left[-\frac{1}{2\kappa^2}R+\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi+V(\Phi)\right]-\frac{1}{2\kappa^2}\int_{\partial M}d^dx\sqrt{\gamma}2K\,, \tag{2.1}$$

where ∂M is the conformal boundary of the (locally) asymptotically AdS spacetime M. By choosing a suitable gauge the bulk metric can be taken to be of the form

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=dr^2+\gamma_{ij}(r,x)dx^idx^j\,, \tag{2.2}$$

where $i=1,\ldots d$. Using the Gauss-Codazzi relations the bulk equations of motion are reduced to [9]

$$\begin{aligned}
K^2-K_{ij}K^{ij}&=R+\kappa^2\left[\dot{\Phi}^2-\gamma^{ij}\partial_i\Phi\partial_j\Phi-2V(\Phi)\right],\\
\nabla_iK^i_j-\nabla_jK&=\kappa^2\dot{\Phi}\partial_j\Phi\,,\\
\dot{K}^i_j+KK^i_j&=R^i_j-\kappa^2\left[\partial^i\Phi\partial_j\Phi+\frac{2}{d-1}V(\Phi)\delta^i_j\right].
\end{aligned} \tag{2.3}$$

Here $K_{ij}=\frac{1}{2}\dot{\gamma}_{ij}$ is the extrinsic curvature and $\dot{K}^i_j$ stands for $\frac{d}{dr}(\gamma^{ik}K_{kj})$. Additionally, we have the equation of motion for the scalar field

$$\ddot{\Phi}+K\dot{\Phi}+\Box\Phi-V'(\Phi)=0 \tag{2.4}$$

and the on-shell action takes the form [9]

$$S_{\mathrm{on-shell}}=-\frac{1}{\kappa^2}\int_{\Sigma_r}d^dx\sqrt{\gamma}(K-\lambda)\,, \tag{2.5}$$

where λ satisfies the equation

$$\dot{\lambda}+K\lambda+\frac{2\kappa^2}{d-1}V(\Phi)=0\,. \tag{2.6}$$

The hamiltonian formalism allows one to write the canonical momenta on a regulating surface Σ_r, diffeomorphic to the boundary ∂M at $r\to\infty$, as functional derivatives of the on-shell action on Σ_r w.r.t. the corresponding bulk field. This shows that the canonical momenta (on Σ_r) are covariant functionals of the induced fields on Σ_r, which in turn leads to an expression for the radial derivative in terms of functional derivatives:

$$\partial_r=\int d^dx2K_{ij}[\gamma,\Phi]\frac{\delta}{\delta\gamma_{ij}}+\int d^dx\dot{\Phi}[\gamma,\Phi]\frac{\delta}{\delta\Phi}\,. \tag{2.7}$$

The asymptotic behavior of the bulk fields then implies that as $r\to\infty$

$$\partial_r\sim\int d^dx2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}+\int d^dx(\Delta-d)\Phi\frac{\delta}{\delta\Phi}\equiv\delta_D\,, \tag{2.8}$$

which is precisely the dilatation operator. This observation is at the center of our approach.

The next step is to expand the canonical momenta and the on-shell action in eigenfunctions of the dilatation operator as

$$\begin{aligned} K^i_j[\gamma,\Phi] &= K_{(0)}{}^i_j + K_{(2)}{}^i_j + \cdots + K_{(d)}{}^i_j + \tilde{K}_{(d)}{}^i_j \log \mathrm{e}^{-2r} + \cdots\,, \\ \lambda[\gamma,\Phi] &= \lambda_{(0)} + \lambda_{(2)} + \cdots + \lambda_{(d)} + \tilde{\lambda}_{(d)} \log \mathrm{e}^{-2r} + \cdots\,, \\ \pi[\gamma,\Phi] &= \dot{\Phi}[\gamma,\Phi] = \sum_{d-\Delta\leq s<\Delta} \pi_{(s)} + \pi_{(\Delta)} + \tilde{\pi}_{(\Delta)} \log \mathrm{e}^{-2r} + \cdots\,, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \delta_D K_{(n)}{}^i_j &= -nK_{(n)}{}^i_j\,, \quad n<d\,, \qquad \delta_D \tilde{K}_{(d)}{}^i_j = -d\tilde{K}_{(d)}{}^i_j\,, \\ \delta_D K_{(d)}{}^i_j &= -dK_{(d)}{}^i_j - 2\tilde{K}_{(d)}{}^i_j\,, \\ \delta_D\lambda_{(n)} &= -n\lambda_{(n)}\,, \quad n<d\,, \qquad \delta_D\tilde{\lambda}_{(d)} = -d\tilde{\lambda}_{(d)}\,, \\ \delta_D\lambda_{(d)} &= -d\lambda_{(d)} - 2\tilde{\lambda}_{(d)}\,, \\ \delta_D\pi_{(s)} &= -s\pi_{(s)}\,, \quad d-\Delta\leq s<\Delta\,, \qquad \delta_D\tilde{\pi}_{(\Delta)} = -\Delta\tilde{\pi}_{(\Delta)}\,, \\ \delta_D\pi_{(\Delta)} &= -\Delta\pi_{(\Delta)} - 2\tilde{\pi}_{(\Delta)}\,. \end{aligned} \tag{2.10}$$

In particular, the coefficients of the logarithmic terms are related to the conformal anomaly,[1] while the terms which transform inhomogeneously under the dilatation operator are in general non-local and correspond to the exact one-point functions of the dual operator. More precisely, we have

$$\langle T_{ij}\rangle_{\rm ren} = -\frac{1}{\kappa^2}\left(K_{(d)ij} - K_{(d)}\gamma_{ij}\right), \qquad \langle\mathcal{O}\rangle_{\rm ren} = \frac{1}{\sqrt{\gamma}}\pi_{(\Delta)}\,. \tag{2.11}$$

At the same time, the renormalized on-shell action is given by

$$S_{\rm ren} = -\frac{1}{\kappa^2}\int_{\Sigma_r} d^dx\sqrt{\gamma}(K_{(d)} - \lambda_{(d)})\,. \tag{2.12}$$

The second equation in (2.3), then leads to the Ward identity

$$\nabla_i\langle T^i_j\rangle_{\rm ren} = -\langle\mathcal{O}\rangle_{\rm ren}\partial_j\Phi\,, \tag{2.13}$$

whereas the following relation, which was derived in [9],

$$(1+\delta_D)K_{(d)} + \kappa^2(\Delta-d)\pi_{(\Delta)}\Phi = (d+\delta_D)\lambda_{(d)} \tag{2.14}$$

gives the (anomalous) Ward identity

$$\langle T^i_i\rangle_{\rm ren} = (\Delta-d)\Phi\langle\mathcal{O}\rangle_{\rm ren} + \mathcal{A}\,. \tag{2.15}$$

[1]Recall that the gravitational part of the conformal anomaly is non-zero only for even dimension d but matter conformal anomalies [16] are generically non-zero for all d. Correspondingly, the logarithmic terms for pure gravity are non-zero only for even boundary dimension d, but they are generically non-zero even for odd d when additional fields are turned on [3].

Here $\mathcal{A} = -\frac{2}{\kappa^2}(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)})$ is the conformal anomaly defined by

$$\delta_D S_{\rm ren} = -\int_{\Sigma_r} d^d x \sqrt{\gamma} \mathcal{A} \,. \tag{2.16}$$

As a consistency check, notice that the above transformation rules (2.10) imply that the conformal anomaly is conformally invariant as required:

$$\delta_D^2 S_{\rm ren} = 0 \,. \tag{2.17}$$

A general algorithm for the recursive evaluation of the coefficients in the covariant expansions was given in [9]. It leads to general expressions for covariant counterterms in arbitrary dimension for pure gravity, but in general the details depend on the matter that couples to gravity and so we will skip the details except when necessary to illustrate the examples we give below. A final comment is due regarding the renormalization scheme dependence of the correlators we have computed. One could add extra finite covariant boundary terms in the action (2.1) without altering the equations of motion. The corresponding terms in the covariant expansions would then be shifted accordingly and so would be the one-point functions. This is all in complete agreement with our expectations from the field theory point of view.

3. Poincaré domain walls

In this section we will consider linear fluctuations around Poincaré domain wall solutions of the equations of motion. These take the generic form

$$ds_B^2 = dr^2 + e^{2A(r)} dx^i dx^i \,, \qquad \Phi = \phi_B(r) \,. \tag{3.1}$$

Inserting this ansatz into the equations of motion we find that $A(r)$ and $\phi_B(r)$ satisfy

$$\dot{A}^2 - \frac{\kappa^2}{d(d-1)}\left(\dot{\phi}_B^2 - 2V(\phi_B)\right) = 0 \,, \tag{3.2}$$

$$\ddot{A} + d\dot{A}^2 + \frac{2\kappa^2}{d-1} V(\phi_B) = 0 \,, \tag{3.3}$$

$$\ddot{\phi}_B + d\dot{A}\dot{\phi}_B - V'(\phi_B) = 0 \,. \tag{3.4}$$

Moreover, one has

$$\dot{\lambda}_B + d\dot{A}\lambda_B + \frac{2\kappa^2}{d-1} V(\phi_B) = 0 \,, \tag{3.5}$$

which immediately implies

$$\lambda_B = \dot{A} + \xi(x) e^{-dA} = \frac{1}{d} K_B + \frac{\xi(x)}{\sqrt{\gamma_B}} \,, \tag{3.6}$$

where ξ is an arbitrary integration function of the transverse coordinates. Therefore,

$$S^B_{\rm on-shell} = -\frac{d-1}{d\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma_B} K_B + \frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \xi(x) \,. \tag{3.7}$$

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The last term corresponds to finite local counterterms. As expected, there is an ambiguity on the on-shell value of the action corresponding to the renormalization scheme dependence of the dual field theory.

It is well known that the second order flat domain wall equations are solved by any solution of the first order flow equations

$$\begin{aligned}\dot{A} &= -\frac{\kappa^2}{d-1} W(\phi_B)\,, \\ \dot{\phi}_B &= W'(\phi_B)\,,\end{aligned} \tag{3.8}$$

provided the potential can be written in the form

$$V(\phi_B) = \frac{1}{2}\left[W'^2 - \frac{d\kappa^2}{d-1}W^2\right]. \tag{3.9}$$

The motivation for considering theories with such potentials stems from the fact that they guarantee gravitational stability of the AdS critical point and of associated domain-wall spacetimes, provided the AdS critical point is also a critical point of W [17, 18, 15]. This means, by the AdS/CFT duality, that the dual theory is unitary. Notice that in principle it is always possible to write the potential in the form (3.9) if one views (3.9) as a differential equation for $W(\phi_B)$. The resulting W however may not have the original AdS spacetime as a critical point. Furthermore, in practice it is considerably difficult to solve (3.9). As a first step in this direction we observe that (3.9) can be transformed into Abel's equation [9]. In general, the set of solutions of the second order equations may include solutions which cannot be obtained from the first order flow equations. In this section we will restrict attention to solutions which can be derived from the flow equations. For this class of solutions we have

$$S^B_{\mathrm{on-shell}} = \int_{\Sigma_r} d^dx\sqrt{\gamma_B}W(\phi_B) + \frac{1}{\kappa^2}\int_{\Sigma_r} d^dx\xi(x)\,. \tag{3.10}$$

3.1 Deformations vs VEVs

The domain wall solutions we have described correspond via the AdS/CFT duality to deformations of the boundary CFT by relevant operators, in which case the conformal invariance of the boundary theory is explicitly broken, or to a vacuum expectation value of a scalar operator which spontaneously breaks the conformal symmetry. Marginal deformations are also similarly described.

Locally asymptotically AdS metrics satisfy the asymptotic condition $\gamma_{ij}(r,x) \sim e^{2r}\gamma_{ij\,(0)}(x)$ as $r \to \infty$, which requires $A(r) \sim r$. Moreover, a scalar field dual to an operator of dimension Δ behaves asymptotically as $\Phi(r,x) \sim e^{-(d-\Delta)r}\phi_{(0)}(x)$. These asymptotic conditions, together with the equations of motion, require that the scalar potential takes the form

$$V(\Phi) = -\frac{d(d-1)}{2\kappa^2} + \frac{1}{2}m^2\Phi^2 + \cdots \tag{3.11}$$

where the mass is related to the dimension Δ of the dual operator by $m^2 = \Delta(\Delta - d)$. Solving the equation of motion for the scalar field with such a potential leads to a generic

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solution of the form

$$\Phi(r,x) = e^{-(d-\Delta)r}\left[\phi_{(0)}(x) + \cdots\right] + e^{-\Delta r}\left[\phi_{(2\Delta-d)}(x) + \cdots\right]. \tag{3.12}$$

We will consider operators for which $d - \Delta \geq \Delta$, or $\Delta \geq d/2$. Moreover, we are interested in relevant or marginal operators and so $\Delta \leq d$. In total then $d/2 \leq \Delta \leq d$. In this range the first term in the solution for Φ is dominant asymptotically and corresponds to the source, while the second term is related to the one-point function of the dual operator. In the special case $\Delta = d/2$, which saturates the BF bound, the scalar field takes the form[2]

$$\Phi(r,x) = e^{-dr/2}\left[-2r\left(\phi_{(0)}(x) + \ldots\right) + \tilde{\phi}_{(0)}(x) + \ldots\right]. \tag{3.13}$$

Again, the first term is the source for the dual operator, while the second is related to its expectation value.

Depending on the form of the 'superpotential' $W(\phi_B)$ the domain wall solution can describe either a deformation of the dual CFT or a phase with spontaneously broken conformal symmetry as we now explain. A similar analysis can be found in [12]. Assuming the potential has a critical point at $\phi_B = 0$, equation (3.9) together with the flow equations (3.8) and the requirement that $\phi_B = 0$ is also a critical point of W, imply that W has an expansion around $\phi_B = 0$ of two possible forms:

$$W_+(\phi_B) = -\frac{d-1}{\kappa^2} - \frac{1}{2}(d-\Delta)\phi_B^2 + \cdots \tag{3.14}$$

$$W_-(\phi_B) = -\frac{d-1}{\kappa^2} - \frac{1}{2}\Delta\phi_B^2 + \cdots. \tag{3.15}$$

Which of these cases is realized is purely a property of the background solution and we need to examine each case separately. Moreover, the extremal values $d/2$ and d of the scaling dimension Δ require special attention. We will now compute the vev for the stress energy tensor and dual scalar operator for all cases.

As was shown in [9], the part of the divergent part of the on-shell action involving only the scalar field is a function $U(\Phi)$, satisfying the equation for the 'superpotential' W (3.9), *and* having an expansion around $\Phi = 0$

$$U(\Phi) = -\frac{d-1}{\kappa^2} - \frac{1}{2}(d-\Delta)\Phi^2 + \cdots \tag{3.16}$$

Let us consider first the case W_+ is realized in the background. In this case we can choose a scheme where the counterterm action is W_+. To see this notice that any two solutions of (3.9) with identical expansions around $\Phi = 0$ up to order Φ^2 can only differ at order $\Phi^{d/(d-\Delta)} \sim e^{-dr}$. This is easily proved by looking for the most general power series[3] solution of (3.9) with this particular form up to quadratic order in Φ. One finds that all terms are uniquely determined up to order $d/(d-\Delta)$ where the recursion relations break

[2] Note that the radial coordinate we use here is related to the Fefferman-Graham radial coordinate by $\rho = e^{-2r}$. The factor $-2r = \log\rho$ is chosen to match with the corresponding formulae in [4, 5].

[3] As usual one must include a logarithmic term at order $d/(d-\Delta)$ in the general case.

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down. This is precisely where an arbitrary integration constant appears and the two solutions could be potentially different. However, this is irrelevant for the purpose of removing the divergences of the on-shell action and so we can choose the renormalization scheme $U(\Phi) = W_+(\Phi)$, and set the integration function ξ to zero. This choice of counterterms corresponds to a supersymmetric renormalization scheme since it ensures $S^B_{\rm ren} = 0$ [4]. It follows that the background vevs of both the operator dual to Φ and the stress tensor vanish identically and so this background describes a deformation of the boundary CFT by a relevant operator.

Let us now consider the case W_- is realized in the background. In this case we cannot choose W_- as the counterterm since it differs from $U(\Phi)$ at the quadratic order. In this case, however, $\phi_B \sim e^{-\Delta r}$ and so the on-shell action evaluated on the background contains only the volume divergence since, by the hypothesis, $2\Delta > d$. Hence, again, setting $\xi = 0$ corresponds to a supersymmetric renormalization scheme with $S^B_{\rm ren} = 0$. Accordingly, the background expectation value of the stress tensor vanishes, but not that of the scalar operator. In this case $W'_-(\phi_B) - U'(\phi_B) = -(2\Delta - d)\phi_B + \ldots$, and hence,

$$\langle \mathcal{O} \rangle^B_{\rm ren} = (d - 2\Delta)\phi_B\,, \tag{3.17}$$

which spontaneously breaks the conformal symmetry of the dual CFT. Here we used the fact that the regularized 1-point function (=canonical momentum) is related to the superpotential via the first order equation (3.8). The renormalized 1-point function is obtained by subtracting the contribution U' of the counterterm.

It remains to examine the two extremal cases $\Delta = d/2$ and $\Delta = d$. When $\Delta = d/2$ there is no distinction between W_+ and W_- as they are equal and $\phi_B \sim e^{-dr/2}$, i.e. it behaves asymptotically as the vev term in (3.13). However, the on-shell action function $U(\Phi)$ includes divergences coming from the source term in (3.13) and therefore it cannot be identified with $W(\Phi)$. It is straightforward to find the covariant counterterms for this case with the method of [9], but the singularity structure for the terms involving the scalar field are not the standard ones, as we now explain. This can be traced back to the fact that the source term for the scalar contains a logarithm, in contrast to the generic case. A simple calculation using the asymptotic form of the solution shows that the canonical momentum of the scalar field takes the form

$$\pi = \dot{\Phi} = \left(\frac{1}{r} - \frac{d}{2}\right)\Phi + \cdots \tag{3.18}$$

Hence, the on-shell action is

$$S_{\rm on-shell} = S^{\rm gr}_{\rm on-shell} + \int_{\Sigma_r} d^dx\sqrt{\gamma}\frac{1}{2}\left(\frac{1}{r} - \frac{d}{2}\right)\Phi^2 + \cdots \tag{3.19}$$

where $S^{\rm gr}_{\rm on-shell}$ is the on-shell action for pure gravity. We have therefore shown that

$$U(\Phi) = -\frac{d-1}{\kappa^2} + \frac{1}{2}\left(\frac{1}{r} - \frac{d}{2}\right)\Phi^2 + \cdots \tag{3.20}$$

which is the divergent part of the on-shell action and must be removed (of course there is an other part coming from pure gravity). The same counterterms (for $d = 4$) were derived in [4, 5]. We now see that the scalar operator gets a vev since $W'(\phi_B) - U'(\phi_B) = -\frac{1}{r}\phi_B + \cdots$, i.e.

$$\langle \mathcal{O} \rangle^B_{\text{ren}} = 2\phi_B \,. \tag{3.21}$$

This also agrees with the results in [4, 5]. Again the difference between $W(\phi_B)$ and $U(\phi_B)$ is subleading and the stress tensor gets no vev since $S_{\text{ren}} = 0$.

Finally we consider the case $\Delta = d$, for which

$$\Phi(r,x) = \left[\phi_{(0)}(x) + \cdots\right] + e^{-dr}\left[\phi_{(d)}(x) + \cdots\right]. \tag{3.22}$$

The equations of motion require $V'(\Phi) = 0$ and so the potential is just the cosmological constant $V(\Phi) = -\frac{d(d-1)}{2\kappa^2}$. It follows that the on-shell function $U(\Phi)$ is also a constant, i.e. the first term of $W_\pm$. In this case, however, the general solution to (3.9) can be easily obtained. There are two distinct solutions (cf. eq. (2.10) in [15]),

$$W_+ = -\frac{d-1}{\kappa^2}, \qquad W_- = -\frac{d-1}{\kappa^2}\cosh\left(\sqrt{\frac{d\kappa^2}{d-1}}(\phi - \phi_o)\right) \tag{3.23}$$

in agreement with our general analysis. Notice that in this case the supergravity action and hence the second order field equations are invariant under constant shifts of the scalar field. Such a constant corresponds to the source term of the solution. One may use this symmetry to set ϕ_0 to zero in W_-. Then, exactly as for the case $d/2 < \Delta < d$, if W_+ is realized in the background, then neither the scalar operator nor the stress tensor acquire a vev, and if W_- is realized, the scalar operator gets a vev $-d\phi_B$, while the vev of the stress energy tensor vanishes.

So finally we can summarize all possibilities for flat domain wall backgrounds in the following table:

Δ	W	$\langle \mathcal{O} \rangle^B_{\text{ren}}$	$\langle T^i_j \rangle^B_{\text{ren}}$
$d/2 < \Delta \leq d$	$+$	0	0
	$-$	$(d-2\Delta)\phi_B$	0
$d/2$	$\pm$	$2\phi_B$	0

3.2 Linearized equations

Now let us consider fluctuations around the backgrounds we have described so far. We will only keep terms up to linear order in fluctuations, which suffices for the calculation of the two point functions. The metric fluctuations take the form

$$\gamma_{ij} = \gamma^B_{ij}(r) + h_{ij}(r,x) = e^{2A(r)}\delta_{ij} + h_{ij}(r,x)\,, \tag{3.24}$$

and the scalar field is

$$\Phi = \phi_B(r) + \phi(r,x)\,. \tag{3.25}$$

The extrinsic curvature then becomes

$$K^i_j = \dot{A}\delta^i_j + \frac{1}{2}\dot{S}^i_j\,, \tag{3.26}$$

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where $S^i_j \equiv \gamma_B^{ik} h_{kj}$. S^i_j can be decomposed into irreducible components as

$$S^i_j = e^i_j + \partial^i \epsilon_j + \partial_j \epsilon^i + \frac{d}{d-1}\left(\frac{1}{d}\delta^i_j - \frac{\partial^i\partial_j}{\Box_B}\right) f + \frac{\partial^i\partial_j}{\Box_B} S\,, \tag{3.27}$$

where $\partial_i e^i_j = e^i_i = \partial_i \epsilon^i = 0$ and indices are raised with the inverse background metric $e^{-2A}\delta^{ij}$. Conversely, each of the irreducible components can be expressed uniquely in terms of S^i_j as

$$e^i_j = \Pi^i{}_k{}^l{}_j S^k_l\,, \qquad \epsilon_i = \pi^l_i \frac{\partial_k}{\Box_B} S^k_l\,, \qquad f = \pi^l_k S^k_l\,, \qquad S = \delta^l_k S^k_l\,, \tag{3.28}$$

where we have introduced the projection operators

$$\Pi^i{}_k{}^l{}_j = \frac{1}{2}\left(\pi^i_k \pi^l_j + \pi^{il}\pi_{kj} - \frac{2}{d-1}\pi^i_j \pi^l_k\right), \tag{3.29}$$

and

$$\pi^i_j = \delta^i_j - \frac{\partial^i\partial_j}{\Box_B}\,. \tag{3.30}$$

With this nomenclature we can now go on and derive the equations of motion for the linear fluctuations. The result is[4]

$$\left(\partial_r^2 + d\dot{A}\partial_r + e^{-2A}\Box\right) e^i_j = 0\,, \tag{3.31}$$

$$\left(\partial_r^2 + [d\dot{A} + 2W\partial_\phi^2 \log W]\partial_r + e^{-2A}\Box\right)\omega = 0\,, \tag{3.32}$$

$$\dot{f} = -2\kappa^2 \dot{\phi}_B \phi\,, \tag{3.33}$$

$$\dot{S} = \frac{1}{(d-1)\dot{A}}\left[-e^{-2A}\Box f + 2\kappa^2\left(\dot{\phi}_B\dot{\phi} - V'(\phi_B)\phi\right)\right] \tag{3.34}$$

where

$$\omega \equiv \frac{W}{W'}\phi + \frac{1}{2\kappa^2} f \tag{3.35}$$

and we have used the diffeomorphism invariance in the transverse space to set $\epsilon_i \equiv 0$. The last two equations give immediately the momenta dual to f and S and hence the corresponding one-point functions with linear sources. Moreover, since the canonical momenta are functionals of the bulk fields [9], to linear order in the fluctuations we must have

$$\dot{e}^i_j = E(A,\phi_B) e^i_j\,, \qquad \dot{\omega} = \Omega(A,\phi_B)\omega\,. \tag{3.36}$$

The first two equations then become first order equations for E and Ω:

$$\begin{aligned} \dot{E} + E^2 + d\dot{A}E - e^{-2A}p^2 &= 0\,, \\ \dot{\Omega} + \Omega^2 + [d\dot{A} + 2W\partial_\phi^2 \log W]\Omega - e^{-2A}p^2 &= 0\,, \end{aligned} \tag{3.37}$$

where we have performed a Fourier transform in the transverse space. Given the solutions for E and Ω we can immediately write down all momenta, namely

$$\dot{e}^i_j = E e^i_j\,, \tag{3.38}$$

[4]Note $\Box_B = e^{-2A}\Box = e^{-2A}\delta^{ij}\partial_i\partial_j$.

$$
\begin{aligned}
\dot{f} &= -2\kappa^2 \dot{\phi}_B \phi \,, && (3.39)\\
\dot{\phi} &= (W'' + \Omega)\phi + \frac{1}{2\kappa^2}\frac{W'}{W}\Omega f \,, && (3.40)\\
\dot{S} &= -\frac{1}{\kappa^2}\left[\left(\frac{W'}{W}\right)^2 \Omega - \frac{e^{-2A}}{W}\Box\right] f - 2\frac{W'}{W}\left(\Omega + \frac{d\kappa^2}{d-1}W\right)\phi. && (3.41)
\end{aligned}
$$

To completely determine the one-point functions with linear sources we first need to obtain exact solutions for E and Ω and secondly, to determine the covariant counterterms for the momenta, *but* only to linear order in the fluctuations. Since e^{-2A} and $W(\phi_B)$ are already covariant functions of the background fields, it suffices to find covariant expansions for E and Ω in the background fields. These can be organized according to the dilatation operator for the background

$$
\delta_D = \partial_A + (\Delta - d)\phi_B \partial_{\phi_B} \,. \tag{3.42}
$$

More generally, the radial derivative is expanded in functional derivatives w.r.t. the background fields as

$$
\partial_r = \dot{A}\partial_A + \dot{\phi}_B \partial_{\phi_B} = -\frac{\kappa^2}{d-1} W(\phi_B)\partial_A + W'(\phi_B)\partial_{\phi_B} \sim \delta_D + \cdots \,. \tag{3.43}
$$

Inserting the following expansions[5] for E and Ω in the first order equations (3.37),

$$
\begin{aligned}
E &= E_{(1)} + \cdots + \tilde{E}_{(d)} \log(e^{-2r}) + E_{(d)} + \cdots \,,\\
\Omega &= \Omega_{(0)} + \cdots + \tilde{\Omega}_{(2\Delta-d)} \log(e^{-2r}) + \Omega_{(2\Delta-d)} + \cdots \,,
\end{aligned} \tag{3.44}
$$

one determines all covariant counterterms which render all momenta finite to linear order in the sources. This procedure is substantially simpler than the general holographic renormalization required to determine the full non-linear counterterms and is a significant improvement over previous methods.

A final simplification can be made for the case of backgrounds corresponding to deformations of the dual CFT. As we saw in the previous section, the 'superpotential' W of the background can be included in the counterterm action, corresponding to a supersymmetric renormalization scheme. After this counterterm is added to the on-shell action (we still have to determine the counterterms for E and Ω), the momenta take the simpler form

$$
\begin{aligned}
\dot{e}^i_j &= E e^i_j \,, && (3.45)\\
\dot{f} &= 0 \,, && (3.46)\\
\dot{\phi} &= \Omega\phi + \frac{1}{2\kappa^2}\frac{W'}{W}\Omega f \,, && (3.47)\\
\dot{S} &= -\frac{1}{\kappa^2}\left[\left(\frac{W'}{W}\right)^2 \Omega - \frac{e^{-2A}}{W}\Box\right] f - 2\frac{W'}{W}\Omega\phi \,. && (3.48)
\end{aligned}
$$

[5]These expansions are strictly correct for $d/2 < \Delta \leq d$, but we will deal with the special case $\Delta = d/2$ in the examples below.

3.3 Examples

We will treat the two examples that have been the main testing ground for holographic computation of correlation functions, namely the GPPZ flow [19] and the Coulomb branch flow [20, 21]. The computation of certain two-point functions for the CB flow was first discussed in [20] and for the GPPZ flow in [22]. Two-point functions for both flows were systematically studied in [23, 4], see also [24, 25, 26] for earlier work. Since the results are known, the emphasis here will be in method rather than the correlators themselves.

GPPZ flow. The GPPZ flow describes a deformation by a supersymmeric mass term of $\mathcal{N}=4$ SYM. The bulk theory is that of a scalar field dual to an operator of dimension $\Delta=3$ coupled to gravity in five dimensions. The background 'superpotential' is

$$W(\phi_B) = -\frac{3}{2\kappa^2}\left[1+\cosh\left(\sqrt{\frac{2}{3}}\kappa\phi_B\right)\right] = -\frac{3}{\kappa^2} - \frac{1}{2}\phi_B^2 + \cdots \tag{3.49}$$

which is of the form W_+ and corresponds to a deformation of the boundary CFT by a relevant operator. The background solution takes the form

$$\phi_B = \frac{1}{\kappa}\sqrt{\frac{3}{2}}\log\left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right), \qquad e^{2A} = \frac{u}{1-u}, \qquad 1-u = e^{-2r}. \tag{3.50}$$

It is also useful to note the relations

$$W = -\frac{3}{\kappa^2}\frac{1}{u}, \qquad W' = -\frac{\sqrt{6}}{\kappa}\frac{\sqrt{1-u}}{u}, \qquad W'' = \frac{2\kappa^2}{3}W + 1. \tag{3.51}$$

Changing variable from r to u in (3.37) we obtain

$$2(1-u)E'(u) + E^2 + \frac{4}{u}E - \frac{1-u}{u}p^2 = 0, \tag{3.52}$$

$$2(1-u)\Omega'(u) + \Omega^2 + \left(\frac{4}{u}-2\right)\Omega - \frac{1-u}{u}p^2 = 0. \tag{3.53}$$

The solutions which are regular at $u=0$ are

$$E(u) = \frac{1}{4}p^2(1-u)\frac{F\left(1-\frac{ip}{2}, 1+\frac{ip}{2}; 3; u\right)}{F\left(-\frac{ip}{2}, \frac{ip}{2}; 2; u\right)} \tag{3.54}$$

and

$$\Omega(u) = \frac{1}{4}p^2(1-u)\frac{F\left(\frac{3-\alpha}{2}, \frac{3+\alpha}{2}; 3; u\right)}{F\left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}; 2; u\right)} \tag{3.55}$$

where $\alpha = \sqrt{1-p^2}$.

Next we need to find covariant counterterms for E and Ω. Inserting

$$\partial_r = \delta_D + \frac{\kappa^2}{6}\phi_B^2\left(\partial_A - \frac{2}{3}\phi_B\partial_{\phi_B}\right) + \cdots \tag{3.56}$$

and the expansions (3.44) in (3.37) one very easily determines

$$E = \frac{p^2}{2} e^{-2A} + \frac{p^2}{4} e^{-2A} \left(\frac{p^2}{2} e^{-2A} + \frac{\kappa^2}{3} \phi_B^2 \right) \log e^{-2r} + E_{(4)} + \cdots , \tag{3.57}$$

$$\Omega = -\frac{p^2}{2} e^{-2A} \log e^{-2r} + \Omega_{(2)} + \cdots . \tag{3.58}$$

Expanding the exact solution in $1-u$ and removing the covariant terms we have just determined allows for the evaluation of $E_{(4)}$ and $\Omega_{(2)}$, which are precisely the terms required to calculate the renormalized one-point functions. Putting everything together we find the following two-point functions:

$$\begin{aligned}
\langle \mathcal{O}(p) \mathcal{O}(-p) \rangle &= -\frac{1}{2} p^2 \bar{J} , \qquad \langle T_i^i(p) \mathcal{O}(-p) \rangle = \frac{\sqrt{6}}{2\kappa} p^2 \bar{J} , \\
\langle T_i^i(p) T_i^i(-p) \rangle &= -\frac{3}{\kappa^2} p^2 (\bar{J} + 1) , \qquad p^j p_i \langle T_j^i \rangle = 0 , \\
\langle T_{ij}(p) T_{kl}(-p) \rangle_{TT} &= \frac{2}{\kappa^2} \Pi_{ijkl} \left[\frac{1}{16} p^2 (p^2 + 4) \bar{K} + \frac{p^2}{8} \right]
\end{aligned}$$

where

$$\begin{aligned}
\bar{J} &= 2\psi(1) - \psi\left(\frac{3}{2} + \frac{1}{2}\sqrt{1-p^2} \right) - \psi\left(\frac{3}{2} - \frac{1}{2}\sqrt{1-p^2} \right) \\
\bar{K} &= \psi(1) + \psi(3) - \psi\left(2 + \frac{ip}{2} \right) - \psi\left(2 - \frac{ip}{2} \right) .
\end{aligned}$$

Coulomb branch flow. The Coulomb branch flow is a solution of five dimensional AdS gravity coupled to a scalar field of mass $m^2 = -4$, which therefore saturates the BF bound. The solution describes the case where an operator of dimension 2 gets a vev. The superpotential is

$$W(\phi_B) = -\frac{2}{\kappa^2} \left[e^{-\kappa\phi_B/\sqrt{3}} + \frac{1}{2} e^{2\kappa\phi_B/\sqrt{3}} \right] . \tag{3.59}$$

The solution can be parametrized by $v \equiv e^{\sqrt{3}\kappa\phi_B}$ as

$$\dot{v} = 2v^{2/3}(1-v) , \qquad e^{-2A} = v^{-2/3}(1-v) , \qquad W = -\frac{1}{\kappa^2} v^{-1/3}(v+2) . \tag{3.60}$$

The boundary is located at $v = 1$. In terms of v the first order equations (3.37) become

$$2(1-v)E'(v) + v^{-2/3}E^2 + \frac{4}{3}\left(1 + \frac{2}{v}\right) E - p^2 v^{-4/3}(1-v) = 0 , \tag{3.61}$$

and

$$2(1-v)\Omega'(v) + v^{-2/3}\Omega^2 + \left[\frac{4}{3}\left(1 + \frac{2}{v}\right) - \frac{12}{v+2} \right] \Omega - p^2 v^{-4/3}(1-v) = 0 . \tag{3.62}$$

The solution for E which is regular at $v = 0$ is

$$E(v) = 2a(1-v)v^{-1/3} \left[1 + \frac{av}{2(a+1)} \frac{F(a+1, a+1; 2a+3; v)}{F(a, a; 2a+2; v)} \right] , \tag{3.63}$$

where $a = -\frac{1}{2} + \frac{1}{2}\sqrt{1+p^2}$. We will not give here explicitly the exact solution for Ω since it is rather complicated. To obtain such a closed form solution one must transform the above equation for Ω into a soluble form and then obtain Ω implicitly through the solution of the transformed equation. After obtaining covariant counterterms for E and Ω by the method we described above, we can write these in the desired form, namely

$$\begin{aligned} E &= \frac{p^2}{2}e^{-2A} + \frac{p^4}{8}e^{-4A}\log e^{-2r} + \frac{p^2}{2}e^{-4A}\left[-\frac{1}{3} + \frac{p^2}{2}\left(\psi(a+1) - \psi(1)\right)\right] + \cdots \qquad (3.64) \\ \Omega &= \frac{1}{r} + \frac{1}{r^2}\left(-\frac{4}{3p^2} + \psi(a+1) - \psi(1)\right) + \cdots . \qquad (3.65) \end{aligned}$$

Inserting these expansions into the expressions for the momenta, after taking into account the effect of the counterterm

$$U(\Phi) = -\frac{3}{\kappa^2} + \frac{1}{2}\left(\frac{1}{r} - \frac{d}{2}\right)\Phi^2 = W(\Phi) + \frac{1}{2r}\Phi^2 + \cdots , \qquad (3.66)$$

we obtain the two-point functions

$$\begin{aligned} \langle \mathcal{O}(p)\mathcal{O}(-p)\rangle &= \left(4\psi(1) - 4\psi(1+a) + \frac{16}{3p^2}\right) \qquad \langle T^i_i(p)\mathcal{O}(-p)\rangle = -\frac{4}{\sqrt{3}\kappa} = 2\langle\mathcal{O}\rangle_B \\ p^ip^j\langle T_{ij}(p)\mathcal{O}(-p)\rangle &= -\frac{2}{\sqrt{3}\kappa}p^2 = \langle\mathcal{O}\rangle_B p^2 \qquad \langle T^i_i(p)T^j_j(-p)\rangle = 0 \\ \langle T_{ij}(p)T_{kl}(-p)\rangle_{TT} &= \Pi_{ijkl}\frac{p^2}{2\kappa^2}\left[\frac{1}{3} - \frac{p^2}{2}\left(\psi(a+1) - \psi(1)\right)\right]. \end{aligned}$$

4. AdS-sliced domain walls

AdS-sliced domain walls have also been studied in the literature [27, 28, 29, 14, 15]. In this case the background is of the form

$$ds_B^2 = dr^2 + e^{2A(r)}g_{ij}(x)dx^idx^i , \qquad \Phi = \phi_B(r) , \qquad (4.1)$$

where $g_{ij}(x)$ is the metric of euclidean AdS_d with radius l and we have set the radius of the bulk AdS_{d+1} equal to 1.[6] Inserting this ansatz into the bulk equations of motion leads to the following equations for $A(r)$ and $\phi_B(r)$

$$\begin{aligned} \dot{A}^2 - \frac{\kappa^2}{d(d-1)}\left(\dot{\phi}_B^2 - 2V(\phi_B)\right) + \frac{1}{l^2}e^{-2A} &= 0 , \qquad (4.2) \\ \ddot{A} + d\dot{A}^2 + \frac{2\kappa^2}{d-1}V(\phi_B) + \frac{d-1}{l^2}e^{-2A} &= 0 , \qquad (4.3) \\ \ddot{\phi}_B + d\dot{A}\dot{\phi}_B - V'(\phi_B) &= 0 . \qquad (4.4) \end{aligned}$$

Note that as $l \to \infty$ these reduce to the equations for flat domain walls, as they should. From now on we set $l^2 = 1$.

[6] In [15] a different convention was used: the radius of bulk AdS_{d+1} and of the AdS_d-slice were set equal to each other.

4.1 Janus solution

A particularly interesting AdS-sliced domain wall solution is the dilaton domain wall solution of type IIB supergravity of [14].[7] This is a non-supersymmetric regular solution. When reduced to five dimensions, it solves the field equations of AdS gravity coupled to a massless scalar with a constant potential. Similar solutions exist in all dimensions [15]. These solutions are of particular interest because they enjoy non-perturbative stability for a broad class of deformations [15]. This strongly suggests that they should have a well-defined QFT dual.

Setting $V = -\frac{d(d-1)}{2\kappa^2}$, the equation for the scalar field can be trivially integrated to give

$$\dot{\phi}_B = ce^{-dA}\,, \tag{4.5}$$

where c is an arbitrary constant of integration. The remaining equations imply,

$$\dot{A}^2 = 1 - e^{-2A} + be^{-2dA} \tag{4.6}$$

where $b = \frac{c^2\kappa^2}{d(d-1)}$. The geometry is non-singular provided the parameter b is within the range,

$$0 \leq b < b_0 \equiv \frac{1}{d}\left(\frac{d-1}{d}\right)^{d-1}. \tag{4.7}$$

One can obtain an implicit solution of (4.6) as

$$r = \int_{A_0}^{A} \frac{dA}{\sqrt{1-e^{-2A}+be^{-2dA}}} \tag{4.8}$$

where A_0 is the smallest zero of $P(u) \equiv bu^d - u + 1$, where $u \equiv e^{-2A}$. This defines half of the geometry, i.e. the region with $0 \leq r < \infty$. The other half is obtained by extending $A(r)$ to negative values of r as an even function, $A(-r) = A(r)$.

We can obtain an explicit expression for the bulk metric by changing variables from r to u. Using

$$\dot{A}^2 = 1 - u + bu^d\,, \tag{4.9}$$

we obtain[8]

$$ds_B^2 = \frac{du^2}{4u^2(1-u+bu^d)} + \frac{1}{u}g_{ij}(x)dx^i dx^i\,. \tag{4.10}$$

Note that if $b = 0$ this is precisely the metric for AdS_{d+1} in the AdS_d-slicing parameterization. The range of the u-coordinate depends on the value of the parameter b, namely $0 \leq u \leq u_o$, where $u_o \geq 1$ with equality iff $b = 0$. We give the explicit form of u_o as a function of b in appendix A. In this parameterization the two halves of the space, i.e. $r > 0$ and $r < 0$, are not distinguished since u is an even function of r. In particular, the regions at $r \to \pm\infty$ are mapped to $u = 0$.

[7] Additional dilatonic deformations have been presented in [30].

[8] An equivalent form of this metric with A instead of u as a variable was found by C. Núñez (unpublished notes, July 2003).

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We discuss in appendix A the conformal compactification of the solution. The conformal boundary consists of two half-spheres with angular excess joined along their equator [14, 15]. We will refer to the joining equator as "corner". In order to calculate correlation functions of the dual field theory we need to write the Janus metric in the Fefferman-Graham (FG) form. Provided the boundary metric is smooth, this is always possible in a neighborhood of the boundary but the FG radial coordinate may in general not be valid far away from the boundary. In the present case, the boundary metric is smooth except for the presence of corners. We therefore except to be able to find a FG coordinates that are well defined in the neighborhood of the boundary except perhaps at the corner.

In appendix B we construct the FG metric to all orders in b for the Janus geometry and determine the range of validity of the radial coordinate. We find that the FG coordinates are well defined everywhere in a neighborhood of the boundary except on the corner where the two half-spheres of the boundary meet. In particular, the FG metric takes the form

$$ds_B^2 = \frac{1}{z_o^2}\left[dz_o^2 + (1 + bc_3(x) + \mathcal{O}(b^2))dz_d^2 + (1 + bc_4(x) + \mathcal{O}(b^2))dz_a^2\right], \tag{4.11}$$

where $x \equiv z_d/z_o$ and z_a, $a = 1, \dots, d-1$ are the standard transverse coordinates in the upper half plane parameterization of the AdS_d slice. The location of the corner is at $z_d = 0$. The functions $c_3(x)$ and $c_4(x)$ as well as the form of the FG metric to all orders in b are given in appendix B. As discussed there this coordinate system covers the region $|x| > x_o = b/\sqrt{2} + \mathcal{O}(b^2)$, so $z_d = 0$ only when $z_o = 0$. In other words, this coordinate system does not cover a (radially extended) neighborhood of $z_d = 0$.

In this coordinate system the background scalar takes the form

$$\phi_B(x) = \phi_o + cc_5(x) + \mathcal{O}(c^3), \tag{4.12}$$

where again $c_5(x)$ is given in the appendix. It is significant to point out here that on the boundary, i.e. $z_o = 0$, the value of the scalar field is a step function in z_d, namely

$$\phi_B(z_d) = \phi_o + \mathrm{sgn}(z_d)c, \tag{4.13}$$

which implies that the coupling of the dual operator is different on the two sides of the corner, or 'wall', at $z_d = 0$. These results are sufficient for calculating correlation functions, which we do in the next section.

4.2 VEVs

Now that we have determined the appropriate FG coordinate system we can carry out the algorithm developed in [9] and evaluate the vevs of the stress tensor and the scalar operator dual to the dilaton, as well as, all two point functions using perturbation theory in c. The first step is to define the radial coordinate[9] $r = -\log z_o$ which is used as the 'time' coordinate in the hamiltonian formalism of [9]. Due to the fact that the background depends

[9]This radial coordinate is different from the original radial coordinate in (4.8) but we hope this causes no confusion.

also on the transverse space coordinates, a full asymptotic analysis is required to determine the covariant counterterms. We will not give these here but they are easily determined following the procedure in [9]. Evaluating these counterterms on the background using the following expressions for the non-vanishing components of the Christoffel symbol and Ricci tensor:

$$\Gamma^d_{dd} = \frac{b}{2}e^r c_3'(x) + \mathcal{O}(b^2)\,, \quad \Gamma^a_{bd} = \frac{b}{2}e^r c_4'(x)\delta^a_b + \mathcal{O}(b^2)\,, \tag{4.14}$$
$$R_{dd} = -\frac{(d-1)b}{2}e^{2r}c_4''(x) + \mathcal{O}(b^2)\,, \quad R_{ad} = R_{ab} = \mathcal{O}(b^2)\,, \quad R = -\frac{(d-1)b}{2}c_4''(x) + \mathcal{O}(b^2)$$

and adding them to the canonical momenta obtained directly by differentiating the background fields w.r.t. r one obtains the following expressions for the vevs of the scalar operator and the stress tensor:

$$\langle\mathcal{O}\rangle_B = c\frac{z_d}{|z_d|^{d+1}}\,, \qquad \langle T^i_j\rangle_B = 0\,. \tag{4.15}$$

Although the calculation has been done to leading order in c, it is not difficult to show that these results are in fact exact. The reason is that the coordinate transformation (A.3) ensures that for every power of b there is a factor of $z_o^{2(d-1)}$ which means that higher order in b terms are subleading and do not survive when the regulator is removed. This can also be seen from the exact expressions for the Fefferman-Graham metric and scalar background given in appendix B, which can be used to obtain the exact canonical momenta. Namely,

$$K^d_{Bd} = \left(1 + \frac{bu^d}{1-u}\right)^{1/2}\,, \qquad K^a_{Bb} = \left(u + \sqrt{(1-u)(1-u+bu^d)}\right)\delta^a_b \tag{4.16}$$

and

$$\dot{\phi}_B(x) = \mathrm{sgn}(x)cu^{d/2}\sqrt{1-u}\,. \tag{4.17}$$

One immediately sees that the vevs given above are in fact exact, as claimed.

The form of the vacuum expectation values is the one required by the symmetries of the problem. As shown in [33], the 1-point functions for a conformal field theory on a flat space with a boundary at $z_d = 0$ (which breaks the conformal group from $O(1, d+1)$ to $O(1, d)$) are precisely of the form (4.15). In the present case we consider the theory on both sides of the wall $z_d = 0$. The McAvity-Osborn result applies separately to the two regions, $z_d > 0$ and $z_d < 0$, and it gives

$$\langle\mathcal{O}\rangle_B = \frac{c_1}{|z_d|^d}\,, \quad z_d > 0\,, \qquad \langle\mathcal{O}\rangle_B = \frac{c_2}{|z_d|^d}\,, \quad z_d < 0\,. \tag{4.18}$$

In the present case $c_1 = -c_2 = c$.

These considerations suggest [14] that the dual field theory for $d = 4$ is $\mathcal{N} = 4$ SYM possibly coupled to non-supersymmetric conformal matter localized at $z_d = 0$ and with g_{YM} being different on the two sides of the wall (similar suggestions can be formulated in all dimensions). This is consistent with the symmetries of the model: the presence of the defect breaks the symmetries to $O(1, 4)$ (i.e. the (euclidean) conformal group in three dimensions). It would be interesting to investigate whether there is a classical solution of $\mathcal{N}=4$ SYM coupled to such defect that can reproduce (4.15), but we will not pursue this here.[10]

4.3 Two-point functions

Since the leading correction to the AdS_{d+1} metric is order c^2, while the leading corrections to the (off diagonal) two-point functions are order c, we can take the background to be exactly AdS and consider linear fluctuations driven by a source $\tilde{T}^i_j$ which is of order c. Decomposing the metric fluctuations as was done for flat domain walls above we derive the following equations for the irreducible components:

$$\begin{aligned} -\Box_g e^i_j &= 2\kappa^2 \Pi^i{}_k{}^l{}_j \tilde{T}^k_l\,, && (4.19)\\ \dot{\epsilon}_j &= 2\kappa^2 \frac{\pi^k_j}{\Box}\tilde{T}_{kd+1}\,, && (4.20)\\ \dot{f} &= -2\kappa^2 \frac{\partial^k}{\Box}\tilde{T}_{kd+1}\,, && (4.21)\\ \dot{S} &= \frac{1}{d-1}\left[2\kappa^2 \tilde{T}_{d+1d+1} - e^{-2r}\Box f\right], && (4.22)\\ -\Box_g\phi &= \frac{1}{2}\dot{\phi}_B\dot{S} - e^{-2r}\left(S^i_j\partial_i\partial^j\phi_B + \partial_i S^i_j\partial^j\phi_B - \frac{1}{2}\partial_j S\partial^j\phi_B\right). && (4.23) \end{aligned}$$

Only the first and the last equations need further analysis as the rest give immediately the momenta as functions of the linear sources. The responses for both the transverse traceless metric fluctuation and the scalar field fluctuation can be obtained using the massless scalar bulk-to-bulk propagator

$$G(\xi) = \frac{c_d}{2^d d}\xi^d F\left(d, \frac{d+1}{2}; \frac{d}{2}+1; \xi^2\right) \qquad (4.24)$$

which satisfies

$$-\Box_g G(\xi) = \delta(z,w) = \frac{1}{\sqrt{g}}\delta(z-w)\,. \qquad (4.25)$$

Here $c_d = \Gamma(d)/(\Gamma(d/2)\pi^{d/2})$ and $\xi = 2z_o w_o/(z_o^2 + w_o^2 + (\vec{z}-\vec{w})^2)$. As $z_o \to 0$

$$G(\xi) \sim \frac{z_o^d}{d}K_d(w,\vec{z}) \qquad (4.26)$$

where

$$K_d(w,\vec{z}) = c_d\left(\frac{w_o}{w_o^2 + (\vec{w}-\vec{z})^2}\right)^d \qquad (4.27)$$

[10] **Note added:** a precise proposal for the dual theory was recently made in [34]: they consider $\mathcal{N}=4$ SYM theory on two half-spaces separated by a planar interface that contains no matter and with a different coupling constant coupled to specific operator closely related to the $\mathcal{N}=4$ lagrangian density. The field theory computations in [34] exactly agree (to the extend that they can be compared) with the holographic computations described in this and next subsection.

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is the well known bulk-to-boundary propagator. To complete the calculation then we need the source $\tilde{T}_{\mu\nu}$ which is

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi \\ &= \partial_\mu\phi_B\partial_\nu\phi + \partial_\nu\phi_B\partial_\mu\phi - g^{AdS}_{\mu\nu}\dot{\phi}_B\left(\dot{\phi} + \frac{e^{-2r}}{z_d}\partial_{z_d}\phi\right) + \mathcal{O}(b)\,. \qquad (4.28)\end{aligned}$$

Here we have used the fact that the background scalar is a function of $x = z_d/z_o$ which implies $z_d\partial_{z_d}\phi_B = \dot{\phi}_B$. With a little more algebra the source can be cast in the form

$$\begin{aligned}\tilde{T}_{d+1d+1} &= \dot{\phi}_B\left(\dot{\phi} - \frac{e^{-2r}}{z_d}\partial_{z_d}\phi\right) + \mathcal{O}(b)\,, \\ \tilde{T}_{jd+1} &= \dot{\phi}_B\left(\partial_j\phi - \frac{1}{z_d}\delta_{jd}\dot{\phi}\right) + \mathcal{O}(b)\,, \\ \tilde{T}_{ij} &= \dot{\phi}_B\left[\frac{1}{z_d}\left(\delta_{id}\partial_j\phi + \delta_{jd}\partial_i\phi - \delta_{ij}\partial_{z_d}\phi\right) - e^{-2r}\dot{\phi}\delta_{ij}\right] + \mathcal{O}(b)\,. \qquad (4.29)\end{aligned}$$

Using these sources and the above bulk-to-bulk propagator we can now evaluate the canonical momenta which give the one-point functions with linear sources. It is not difficult to show that no counterterms contribute to the order c terms of the momenta. It turns out to be easier to obtain the two-point functions from the canonical momentum of the graviton by differentiating w.r.t. the scalar source rather than from the scalar momentum and so we only consider the graviton momentum here. Of course both calculations should give identical results and we show this explicitly in appendix C, where we calculate the two-point functions from the scalar momentum.

To obtain the the canonical momentum of the transverse traceless component of the graviton we note that the inhomogeneous solution to its equation of motion is

$$e^i_j = 2\kappa^2\int d^{d+1}w\sqrt{g(w)}G(\xi)\Pi^i{}_k{}^l{}_j\tilde{T}^k_l(w) \qquad (4.30)$$

so that asymptotically

$$\dot{e}^i_j \sim -2\kappa^2 e^{-dr}\int d^{d+1}w\sqrt{g(w)}K_d(w,\vec{z})\Pi^i{}_k{}^l{}_j\tilde{T}^k_l(w)\,. \qquad (4.31)$$

Substituting the above expressions for the source into the canonical momenta and differentiating w.r.t. the scalar source we arrive at the two-point functions

$$\begin{aligned}\Pi^i{}_k{}^l{}_j\langle T^k_l(\vec{z})\mathcal{O}(\vec{w})\rangle &= -2(d+1)c\Pi^i{}_d{}^d{}_jI(\vec{z},\vec{w})\,, & (4.32)\\ \pi^l{}_j\frac{\partial_k}{\Box}\langle T^k_l(\vec{z})\mathcal{O}(\vec{w})\rangle &= \frac{\pi^k{}_j}{\Box}\left(\langle\mathcal{O}(\vec{z})\rangle_B\partial_k\delta^{(d)}(\vec{z}-\vec{w})\right), & (4.33)\\ \pi^l{}_k\langle T^k_l(\vec{z})\mathcal{O}(\vec{w})\rangle &= -\frac{\partial^k}{\Box}\left(\langle\mathcal{O}(\vec{z})\rangle_B\partial_k\delta^{(d)}(\vec{z}-\vec{w})\right), & (4.34)\\ \langle T^i_i(\vec{z})\mathcal{O}(\vec{w})\rangle &= 0\,, & (4.35)\end{aligned}$$

where the projection operators are acting on $\vec{z}$ and

$$I(\vec{z},\vec{w}) = c_d^2\int d^dx dx_o x_o^{2d+1}\frac{x_d}{(x_o^2+x_d^2)^{(d+3)/2}}\frac{1}{[x_o^2+(\vec{x}-\vec{z})^2]^d}\frac{1}{[x_o^2+(\vec{x}-\vec{w})^2]^d}\,. \qquad (4.36)$$

The last three correlators can be easily seen as a consequence of the Ward identities (2.13) and (2.15), which is a non-trivial consistency check of our calculation. In fact, since the vevs are exact, these two-point functions must also be exact in c, although our calculation of the two-point functions was done only to leading order in c.

The computation of the remaining of the 2-point functions $\langle T_{ij}(x)T_{kl}(y)\rangle$ and $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$ requires an analysis to order c^2. This computation is rather complex since the background metric receives a correction at this order. This means that we need to linearize the bulk field equations around the corrected solution (B.5). The latter, however, is inhomogeneous and this complicates the analysis. Nevertheless, the conformal invariance of the boundary theory completely determines the two point function of the scalar operator, while the two-point function of the stress tensor is determined up to a scalar function (except for $d=2$ where it is fully determined) [33]. It would be interesting to check that the holographic calculation reproduces these two-point functions as well.

4.4 Janus two-point functions vs boundary CFT

Let us now take a closer look at the structure of the Janus two-point functions. We would like to show that the 2-point functions are of the form required by conformal invariance for a CFT on a space with a wall at $z_d = 0$.[11] The subgroup of the conformal group $O(1,d+1)$ that leaves $z_d = 0$ invariant is $O(1,d)$. This is precisely the isometry group of the Janus metric. McAvity and Osborn [33] have given explicitly the form of this two-point function in such a CFT. It is given by

$$\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle = -\mathrm{sgn}(z_d)c\frac{2^{d-1}d^2\Gamma\left(\frac{d}{2}\right)}{(d-1)\pi^{d/2}}\left(\frac{v}{\vec{s}^2}\right)^d\left(X^iX_j - \frac{1}{d}\delta^i_j\right), \tag{4.37}$$

where[12]

$$v^2 = \frac{\xi}{\xi+1}, \qquad \xi = \frac{\vec{s}^2}{4z_dw_d}, \qquad \vec{s} = \vec{z}-\vec{w}, \tag{4.38}$$

and

$$X_i = z_d\frac{v}{\xi}\partial_i\xi = v\left(\frac{2z_d}{\vec{s}^2}s_i - n_i\right), \tag{4.39}$$

with $n_i = \delta_{id}$. The normalization of the two-point function is fixed by the normalization of the vev of the scalar operator [33]. It is clear from (4.18) that this normalization has opposite signs for $z_d > 0$ and $z_d < 0$, which is the origin of the $\mathrm{sgn}(z_d)$ factor. This expression applies for $z_dw_d > 0$, i.e. both points on the same side of the wall, but not for $z_dw_d < 0$. The holographic expression (C.8) applies to both cases, however. Under conformal transformations that leave the hyperplane $z_d = 0$ invariant we have

$$\vec{s}^2 \to \frac{\vec{s}^2}{\Omega(\vec{z})\Omega(\vec{w})}, \qquad z_d \to \frac{z_d}{\Omega(\vec{z})}, \qquad w_d \to \frac{w_d}{\Omega(\vec{w})}. \tag{4.40}$$

[11] We are grateful to the authors of [34] for pointing us to the work of McAvity and Osborn and for prompting us to check that our calculation is consistent with their results.

[12] We use ξ here to conform with the notation of [33]. This should not be confused with the argument of the bulk-to-bulk propagator used earlier.

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It follows that ξ is a conformal invariant while X_i transforms as a vector. In particular, under inversion $\vec{z} \to \vec{z}/\vec{z}^2$, $\vec{w} \to \vec{w}/\vec{w}^2$,

$$X_i \to I_{ij}(\vec{z}) X_j\,, \tag{4.41}$$

where $I_{ij}(\vec{z}) = \delta_{ij} - 2\frac{z_i z_j}{\vec{z}^2}$. It is easy then to see that the two-point function given above transforms correctly under inversion, namely

$$\langle T^i_j(\vec{z'}) \mathcal{O}(\vec{w'})\rangle = \vec{z}^{2d} \vec{w}^{2d} I^i_k(\vec{z}) I^l_j(\vec{z}) \langle T^k_l(\vec{z}) \mathcal{O}(\vec{w})\rangle\,. \tag{4.42}$$

One can show in general, using the fact that the background has the correct isometries, that the holographic 2-point functions transform as they should. Since the results of [33] follow from the same symmetries, this argument shows that our results are consistent with that of [33]. It is, however, a rather non-trivial exercise to explicitly demonstrate that the correlator is of the form given in [33], mainly because of the integral representation of the transverse-traceless part of the correlator. The integral that appears in the transverse traceless part of the holographic two-point function is not easy to evaluate in general, and evaluating the projection operator acting on it is not straightforward either. This makes a direct comparison of the two results rather non-trivial. Instead, we will expand both results in a short distance expansion and compare them term by term. We do this for the first three orders in the expansion and we find complete agreement.

To facilitate the comparison we first expand the above result of McAvity and Osborn. Of course, this expansion is valid only when $z_d w_d > 0$, which is also the condition for the validity of the McAvity-Osborn expression. After some algebra we get

$$\begin{aligned}
\langle T^i_j(\vec{z}) \mathcal{O}(\vec{w})\rangle \;=\; & -c \frac{2^{d-1} d^2 \Gamma(d/2)}{(d-1)\pi^{d/2}} \frac{2w_d}{|2w_d|^{d+1}} \frac{1}{(\vec{s}^2)^{d/2}} \times \\
& \times \Bigg\{ \frac{s^i s_j}{\vec{s}^2} - \frac{1}{d}\delta^i_j - \frac{1}{2w_d}\left(n^i s_j + n_j s^i - \vec{n}\cdot\vec{s}\delta^i_j + (d-2)\vec{n}\cdot\vec{s}\frac{s^i s_j}{\vec{s}^2}\right) + \\
& + \frac{1}{(2w_d)^2}\Bigg[\frac{1}{2}\left(d(d-2)(\vec{n}\cdot\vec{s})^2 - (d+2)\vec{s}^2\right)\frac{s^i s_j}{\vec{s}^2} + 2d(\vec{n}\cdot\vec{s}) n^{(i} s_{j)} + \\
& + \vec{s}^2 n^i n_j - \frac{1}{2}((d+2)(\vec{n}\cdot\vec{s})^2 - \vec{s}^2)\delta^i_j\Bigg] + \mathcal{O}(s^3)\Bigg\}\,.
\end{aligned} \tag{4.43}$$

The holographic result can be easily evaluated to this order too. First, using

$$\delta^{(d)}(\vec{s}) = -\frac{\Gamma\left(\frac{d}{2}\right)}{2(d-2)\pi^{d/2}} \Box \frac{1}{(\vec{s}^2)^{\frac{d-2}{2}}}\,, \tag{4.44}$$

we find that the longitudinal part of the holographic two-point function reproduces precisely the first two orders of the short distance expansion of the McAvity and Osborn result. The transverse traceless part is then evaluated by acting with the projection operator on

$$I(\vec{z},\vec{w}) = \frac{\Gamma\left(\frac{d}{2}-1\right) d^2}{8(d+1)\pi^{d/2}} \frac{w_d}{|w_d|^{d+3}} \frac{1}{(\vec{s}^2)^{\frac{d-2}{2}}} \left(1 + \mathcal{O}(s)\right)\,, \tag{4.45}$$

and it reproduces exactly the third order term. Some details of this calculation are presented in appendix D. Therefore, at least to this order in the short distance expansion, we have shown that the holographic two-point function is exactly what one expects for a CFT with a wall at $z_d = 0$.

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5. Conclusions

We discuss in this paper the computation of correlation functions for holographic RG flows. The computation was done within the hamiltonian framework developed in [9]. A central point in our discussion is that the analysis is focused on the canonical momenta which are associated with regularized correlation functions and not the on-shell action. Furthermore, the renormalization procedure is set up such that one only computes the contribution of counterterms to correlators under discussion rather than first computing the general counterterms and then specializing. In particular, to renormalize n-point functions we only need to obtain asymptotic solutions to $(n-1)$-order in fluctuations. For instance, for 2-point functions a linearized analysis is sufficient. This leads to significant reduction of labor compared to previous works.

In the literature the analysis of fluctuations and of renormalization were often performed in different coordinate systems. This was due to the fact that renormalization was heavily based in the universal form of AdS asymptotics which required the use of a particular coordinate system, the Fefferman-Graham coordinate system. The fluctuation equation however is often more easily solvable in other coordinates. In order to combine the renormalization results with the solution of the fluctuation one needs the transformation between the two coordinate systems. This is straightforward to obtain, at least asymptotically, which is the only thing needed, but it adds to the complexity of the method. In this work we also avoid this complexity, since the asymptotic analysis is done directly at the level of the fluctuation equations. Moreover, it is not necessary to use the Fefferman-Graham coordinates anymore since the asymptotic expansion is now done by organizing the asymptotic solution in terms of eigenfunctions of the dilatation operator [9].

To illustrate the method we discussed both flat and AdS-sliced domain walls. We reduced the computation of 2-point functions that involve the stress energy tensor and the operator dual to the active scalar for the most general flat domain walls driven by a "superpotential" W to the solution of two second order linear ODEs or (equivalently) to two first order non-linear ODEs. With the new method it is easy to carry out the renormalization in general (and we discussed how to do this) but it is even easier — almost trivial — to carry out the procedure in each specific case. For comparison purposes, we discussed all details for the two cases mostly studied in the literature, the GPPZ and CB flows. The new method is comparable in efficiency to the "old approach" where one extracts the correlator from the leading non-analytic part of the linearized solution, but it does not suffer from any of its drawbacks.

As a new example, we discussed the computation of correlation functions of the Janus solution. This is a regular non-supersymmetric but stable AdS-sliced domain wall solution. The main difficulty in this example is that the boundary has a corner. However, we showed that there exists a Fefferman-Graham coordinate system which is well-defined everywhere in the neighborhood of the boundary except on the corner. This allows the vev's of the dual operators to be read off. We found that the expectation value of the stress-energy tensor is zero and the expectation value of the operator dual to the active scalar is non-zero. The expectation value is non-homogeneous and blows up at the location of the corner, which

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plays the role of a 'wall' in the boundary CFT. We computed then the 2-point functions that receive a contribution to leading order in c, i.e. $\langle T_{ij}(x)\mathcal{O}(y)\rangle$. Ward identities relate (part of) $\langle T_{ij}(x)\mathcal{O}(y)\rangle$ to the vev $\langle\mathcal{O}\rangle$. It is a nice check of both the value of the vev and of the 2-point function that the Ward identity is indeed satisfied. Both the vev and the 2-point function we computed are of the form implied by conformal invariance for a CFT on $\mathbf{R}^d$ with a wall at $z_d = 0$.

We discussed here only 2-point functions. Higher point functions have been discussed in the context of holographic renormalization in [7, 35]. The new method can be straightforwardly applied in such cases, essentially trivializing the issue of renormalization. Another direction in which the new method holds promise in delivering new results is the case of solutions that are not asymptotically AdS such as the Klebanov-Strassler solution [36].

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A. Conformal compactification of the Janus solution

In order to determine the conformal compactification of the Janus geometry we introduce a new radial coordinate

$$z = \dot{A} = \pm\sqrt{1 - u + bu^d}\,. \tag{A.1}$$

This coordinate has the range $-1 \leq z \leq 1$ for *any* value of b and the $u = 0$ region is mapped to $z = \pm 1$. u can be determined as a function of z by solving the algebraic equation

$$u - bu^d = 1 - z^2 \tag{A.2}$$

as a power series in b. The relevant solution is the smallest real positive root which is given for arbitrary d and to all orders in b by

$$\begin{aligned} u(z; b, d) &= (1 - z^2)\sum_{n=0}^{\infty}\frac{\Gamma(nd+1)}{\Gamma(n+1)\Gamma(n(d-1)+2)}b^n(1-z^2)^{n(d-1)} \\ &= {}_{(d-1)}F_{(d-2)}\Bigg[\left(\frac{1}{d},\frac{2}{d},\ldots,\frac{d-1}{d}\right),\left(\frac{2}{d-1},\frac{3}{d-1},\ldots,\frac{d-2}{d-1},\frac{d}{d-1}\right), \\ &\qquad \frac{d^d}{(d-1)^{d-1}}b(1-z^2)^{d-1}\Bigg]\,, \end{aligned} \tag{A.3}$$

where ${}_pF_q$ is the generalized hypergeometric function. Note that the bound of the u coordinate mentioned above is just $u_o = u(0; b, d)$. It is not possible to express $u(z)$ in terms of elementary functions except for the cases $d = 2$ and $d = 3$. We have respectively,

$$u(z; b, 2) = \frac{1}{2b}\left(1 - \sqrt{1 - 4b(1 - z^2)}\right), \quad u(z; b, 3) = \frac{2}{\sqrt{3b}}\sin\left[\frac{1}{3}\arcsin\left(\frac{3}{2}\sqrt{3b}(1 - z^2)\right)\right]. \tag{A.4}$$

If we now write the (euclidean) AdS_d-slice metric in global coordinates and set $z = \sin\theta$, the metric (4.10) becomes

$$ds_B^2 = \frac{1}{u(\sin\theta)\cos^2\lambda}\Big[d\lambda^2 + \cos^2\lambda\Big(1 + (2d-1)b(\cos^2\theta)^{d-1} + \mathcal{O}(b^2)\Big)d\theta^2 + \\ + d\tau^2 + \sin^2\lambda d\Omega_{d-2}^2\Big], \tag{A.5}$$

where $0 \le \lambda \le \pi/2$ and $-\pi/2 \le \theta \le \pi/2$. A few comments are in order here. First, the transformation (A.3) implies that every power of b in the coefficient of $d\theta^2$ comes with a factor of $(\cos^2\theta)^{d-1}$ and hence the metric inside the square brackets is non-singular for any θ. Second, note that the (λ, θ) part of the metric can be transformed into the standard metric on $\mathbf{S}^2$ by introducing the angular coordinate

$$\mu = \int_0^{\sin\theta} \frac{dz}{\sqrt{u(z)}\,(1 - bdu(z)^{d-1})} = \theta + \left(d - \frac{1}{2}\right) b\sin\theta F\left(\frac{1}{2}, \frac{3}{2} - d; \frac{3}{2}; \sin^2\theta\right) + \mathcal{O}(b^2)\,. \tag{A.6}$$

This is precisely the angular coordinate introduced in [14, 15] and it takes values in $[-\mu_o, \mu_o]$, where $\mu_o \ge \pi/2$ is given in equation (B.8) of [15]. Because of the excess angle the compact metric has a corner at $\lambda = \pi/2$, as is discussed in [15].

B. Fefferman-Graham coordinates for Janus metric

To construct the Fefferman-Graham metric we start with (4.10) and the coordinate transformation (A.3) and write the AdS_d-slice metric in the upper-half plane coordinates. Then

$$ds_B^2 = \frac{dz^2}{(1-z^2)^2}\left[1 + 2(d-1)b(1-z^2)^{d-1} + \mathcal{O}(b^2)\right] + \\ + \frac{1}{1-z^2}\left[1 - b(1-z^2)^{d-1} + \mathcal{O}(b^2)\right]\frac{1}{\tilde{z}_o^2}(d\tilde{z}_o^2 + dz_a^2), \tag{B.1}$$

where $a = 1, \ldots, d-1$. For $b = 0$ the coordinate transformation

$$z = \frac{z_d}{\sqrt{z_o^2 + z_d^2}}, \qquad \tilde{z}_o = \sqrt{z_o^2 + z_d^2} \tag{B.2}$$

brings this metric into the upper half plane metric with radial coordinate z_o. To determine the Fefferman-Graham form of the Janus metric we need to obtain appropriate b-dependent corrections to this transformation. We can determine these as a Taylor series in b by introducing two arbitrary functions at each order in b and solving the differential equations that result by requiring that the transformed metric is of the Fefferman-Graham form. The unique transformation which ensures that the metric remains asymptotically AdS independent of b is to linear order in b

$$z = \frac{z_d}{\sqrt{z_o^2 + z_d^2}} + bf_1(x) + \mathcal{O}(b^2)\,, \qquad \tilde{z}_o = \sqrt{z_o^2 + z_d^2} + bz_o f_2(x) + \mathcal{O}(b^2)\,, \tag{B.3}$$

where $x \equiv z_d/z_o$ and

$$
\begin{aligned}
f_1(x) \;&=\; \frac{x}{2(1+x^2)^{3/2}}\Bigg[\left(1-\frac{1}{2d}\right)\frac{1}{x^{2d}}F\left(d,d;d+1;-\frac{1}{x^2}\right)+\\
&\qquad +\frac{1}{2(d+1)x^{2(d+1)}}F\left(d+1,d;d+2;-\frac{1}{x^2}\right)+\frac{1}{(1+x^2)^{d-1}}\Bigg]\,,\\
f_2(x) \;&=\; \frac{1}{2\sqrt{1+x^2}}\Bigg[\frac{1+2dx^2}{2dx^{2d}}F\left(d,d;d+1;-\frac{1}{x^2}\right)+\\
&\qquad +\frac{1}{2(d+1)x^{2d}}F\left(d+1,d;d+2;-\frac{1}{x^2}\right)-\frac{1}{(1+x^2)^{d-1}}\Bigg]\,. \qquad \text{(B.4)}
\end{aligned}
$$

The metric then takes the from

$$
ds_B^2 = \frac{1}{z_o^2}\left[dz_o^2 + (1+bc_3(x)+\mathcal{O}(b^2))dz_d^2 + (1+bc_4(x)+\mathcal{O}(b^2))dz_a^2\right], \qquad \text{(B.5)}
$$

where

$$
\begin{aligned}
c_3(x) \;&=\; \frac{(2d-1)}{2dx^{2d}}F\left(d,d;d+1;-\frac{1}{x^2}\right)+\frac{1}{2(d+1)x^{2(d+1)}}F\left(d+1,d;d+2;-\frac{1}{x^2}\right)-\\
&\qquad -\frac{1}{x^2(1+x^2)^{d-1}}\,,\\
c_4(x) \;&=\; -\frac{1}{2dx^{2d}}F\left(d,d;d+1;-\frac{1}{x^2}\right). \qquad \text{(B.6)}
\end{aligned}
$$

Note that the derivatives of these functions have a much simpler form:

$$
c_3'(x) = \frac{1}{x^3(1+x^2)^d}\left[-1-(2d-1)x^2+2\frac{1+(d+2)x^2}{(1+x^2)^2}\right], \qquad c_4'(x) = \frac{1}{x(1+x^2)^d}\,. \qquad \text{(B.7)}
$$

The metric in (B.5) is manifestly invariant under translations and rotations of the z_a coordinates and scale transformations (the x coordinate is invariant under scale transformations). The original metric (4.10) however was invariant under the larger group $O(1,d)$ associated with the AdS slice metric. We now show that the metric (B.5) is also invariant under a discrete inversion isometry to order b which enhances the isometry group to the full $O(1,d)$. Actually, we will see that the inversion symmetry can be used to obtain the Fefferman-Graham form of the metric to all orders in b.

Let us write the AdS slice metric in (4.10) in the upper half plane coordinates so that

$$
ds_B^2 = \frac{du^2}{4u^2(1-u+bu^d)} + \frac{1}{u\tilde{z}_o^2}(d\tilde{z}_o^2 + dz^a dz^a)\,. \qquad \text{(B.8)}
$$

This form is invariant under the discrete isometry

$$
\tilde{z}_o \to \frac{\tilde{z}_o}{\tilde{z}_o^2+z_a^2}\,, \qquad z_a \to \frac{z_a}{\tilde{z}_o^2+z_a^2}\,. \qquad \text{(B.9)}
$$

We now bring this metric into the Fefferman-Graham form by means of a coordinate transformation[13]

$$
z = s(x;b)\,, \qquad \tilde{z}_o = z_o t(x;b)\,, \qquad \text{(B.10)}
$$

[13]Note that $u = u(z;b)$ is given in eq. (A.3).

where $x = z_d/z_o$. We point out that this is precisely the form of the coordinate transformation (B.3), but we now treat the b-dependence non-perturbatively. This allows us to express the above discrete isometry in terms of the Fefferman-Graham coordinates $z^\mu = (z_o, z_a, z_d)$. We find

$$z^\mu \to \frac{z^\mu}{z_o^2 t(x;b)^2 + z_a^2} \,. \tag{B.11}$$

Now, the Fefferman-Graham metric (B.5) takes the form

$$ds_B^2 = \frac{1}{z_o^2}\left[dz_o^2 + \lambda(x;b)dz_d^2 + \mu(x;b)dz_a^2\right] . \tag{B.12}$$

The requirement that this is invariant under inversion uniquely determines the functions $\lambda(x;b)$ and $\mu(x;b)$ in terms of $t(x;b)$. Namely we find the exact FG metric

$$\boxed{ds_B^2 = \frac{1}{z_o^2}\left[dz_o^2 + \frac{\partial_x t}{x(t - x\partial_x t)}dz_d^2 + \frac{1}{t(t - x\partial_x t)}dz_a^2\right] .} \tag{B.13}$$

Requiring further that this is equal to the Janus metric above uniquely fixes the transformation functions $s(x;b)$ and $t(x;b)$. In particular we obtain the system of coupled equations

$$u = 1 - x\frac{\partial_x t}{t} \,, \qquad (\partial_x s)^2 = \frac{1}{x^2}u^2(1-u)(1 - bdu^{d-1})^2 \,, \tag{B.14}$$

where $u = u(s(x))$ is given by (A.3).

In order to solve these equations we use (A.1) to trade $s(x)$ for $u(x)$ in the second equation, which gives

$$\boxed{\int^{u(x)} \frac{du'}{u'\sqrt{(1-u')(1-u'+bu'^d)}} = -\log x^2 \,.} \tag{B.15}$$

The sign and the integration constant are chosen so that $u(x) \sim 1/x^2$ as $x \to \infty$, independent of b. Unfortunately it seems rather difficult to do this integral explicitly for arbitrary dimension d. Instead, one can expand the integrand in b and integrate term by term. This gives

$$u(x) = \frac{1}{1+x^2} + \frac{b}{2d}\frac{x^2}{(1+x^2)^{d+2}}F\left(d, 2; d+1; \frac{1}{1+x^2}\right) + \mathcal{O}(b^2) \,. \tag{B.16}$$

The transformation functions $s(x;b)$ and $t(x;b)$ are now determined from

$$\boxed{\begin{aligned} s(x) &= 1 - u(x) + bu(x)^d \\ t(x) &= \exp\left[\frac{1}{2}\int_{u(x)}^1 \frac{du'}{u'}\left(\frac{1-u'}{1-u'+bu'^d}\right)^{1/2}\right] . \end{aligned}} \tag{B.17}$$

Inserting the above expansion for $u(x)$ we reproduce (after some manipulation of the hypergeometric functions) precisely the coordinate transformation (B.3).

Moreover, inserting (B.16) in

$$\boxed{\phi_B(x) = \phi_o + c\int_0^x \frac{dx'}{|x'|}u(x')^{d/2}\sqrt{1-u(x')}} \tag{B.18}$$

gives

$$\phi_B(x) = \phi_o + c c_5(x) + \mathcal{O}(c^3)\,, \tag{B.19}$$

where ϕ_o is a constant and

$$c_5(x) = \frac{x}{\sqrt{1+x^2}} F\left(\frac{1}{2}, 1-\frac{d}{2}; \frac{3}{2}; \frac{x^2}{1+x^2}\right). \tag{B.20}$$

Again, this has a simple derivative:

$$c_5'(x) = \frac{1}{(1+x^2)^{(d+1)/2}}\,. \tag{B.21}$$

Notice that as $z_o \to 0$ with all other coordinates fixed (i.e. as we approach the conformal boundary) $c_5(x) = \mathrm{sgn}(z_d)$ while higher order terms do not contribute. So at the boundary

$$\phi_B(z_d) = \phi_o + \mathrm{sgn}(z_d) c\,. \tag{B.22}$$

This implies that the coupling of the dual operator is different on the two sides of the wall.

Finally, let us examine the range of validity of the coordinate transformation (B.10). The jacobian of the transformation is equal to $J = t\partial_x s$. Now $J = 0$ implies $\partial_x s = 0$ since $t(x)$ is positive definite, as can be seen from (B.17). It follows that the coordinate transformation breaks down at $u = 1$. Note that the zero of $(1 - bdu^{d-1})$ occurs at $u = 1/(bd)^{1/(d-1)} > 1$, where the inequality follows from (4.7). We conclude that the Fefferman-Graham coordinates are valid in the range $0 < u < 1$ although, in general $0 \leq u \leq u_o$ with $u_o \geq 1$. Recall that in general the Fefferman-Graham coordinate system is only guaranteed to exist in a neighborhood of the boundary, and here we see an explicit illustration of this.

Recall that the Fefferman-Graham coordinate system [31] is obtained as follows (see section 3 of [7] for a review). One considers gaussian normal coordinates centered at the boundary and the radial coordinate is identified with the affine parameter of the geodesics emanating perpendicularly from the boundary. Clearly the region of validity of this coordinate system depends on the behavior of the radial geodesics. We therefore need to analyze such geodesics, and we will do this in the $(u, \tilde{z}_o, z_a)$ coordinate system which is well-defined everywhere.

One easily shows that there are geodesics with z_a constant. The geodesic equations for the remaining coordinates lead to the following two equations

$$\frac{d\log\tilde{z}_o}{d\tau} = a_1 u\,, \tag{B.23}$$

$$\ddot{u} - \left(\frac{1}{u} + \frac{-1+bdu^{d-1}}{2(1-u+bu^d)}\right)\dot{u}^2 + 2a_1^2 u^2(1-u+bu^d) = 0\,, \tag{B.24}$$

where a_1 is an integration constant. If $a_1 \neq 0$ the second equation can be integrated once to get

$$\dot{u} = \pm 2a_1 u\sqrt{(a_2-u)(1-u+bu^d)} \tag{B.25}$$

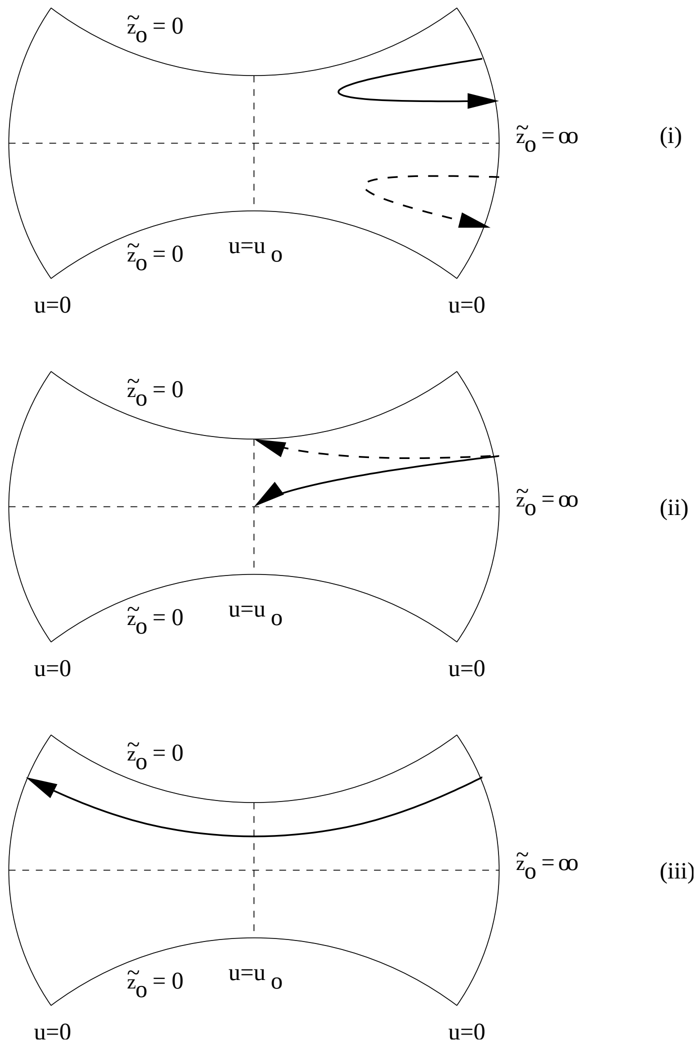

Figure 1: The qualitatively different radial geodesics of the Janus geometry: (i) $a_2 < u_o$ with $a_1 > 0$ (solid arrow) or $a_1 < 0$ (broken arrow), (ii) $a_2 = u_o$ with $a_1 > 0$ (solid arrow) or $a_1 < 0$ (broken arrow) and (iii) $a_1 = 0$. These qualitative features are insensitive to the value of b.

for some constant $0 < a_2 \leq u_o$. If $a_1 = 0$ one gets instead

$$\dot{u} = \pm a_3 u \sqrt{1 - u + bu^d}\,, \tag{B.26}$$

where a_3 is again a constant. Now, depending on the values of the parameters a_1 and a_2, we can identify three qualitatively different types of geodesics as shown in figure 1.

Consider now the radial geodesics defined by $\dot{z}_d = \dot{z}_a = 0$ in the Fefferman-Graham coordinates, where the dot stands for the derivative w.r.t. the affine parameter $\tau = \log z_o$. Since $z_d =$ constant along these geodesics we will take $\tau = \log(z_o/|z_d|) = -\log|x|$ for later convenience. The transformation (B.10) immediately gives

$$\frac{d \log \tilde{z}_o}{d\tau} = u\,, \tag{B.27}$$

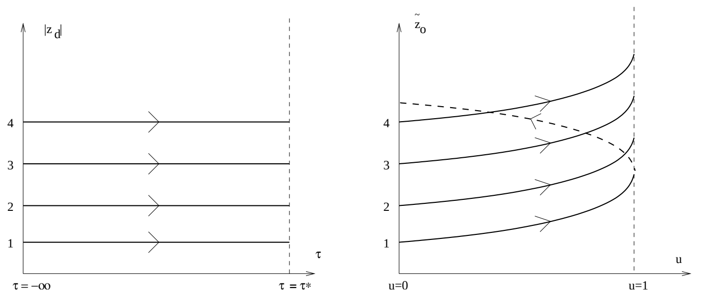

Figure 2: A radial geodesic in the Fefferman-Graham coordinates (left) is not defined beyond $u = 1$ due to the failure of the coordinate system at this point. In the coordinate system which extends beyond $u = 1$ this geodesic bounces back at $u = 1$. Only the branch before the bounce corresponds to the geodesic on the left.

while (B.15) implies

$$\dot{u} = 2u\sqrt{(1-u)(1-u+bu^d)}\,. \tag{B.28}$$

The Fefferman-Graham radial geodesics therefore correspond to radial geodesics with $a_1 = a_2 = 1$. In particular, they are geodesics of type (i) if $b > 0$ but they are type (ii) if $b = 0$. This is an important qualitative difference between the FG coordinates for the Janus geometry and pure AdS. This is in fact why the FG coordinates cover the whole of AdS but only part of the Janus geometry.

It is now clear why the FG coordinate system for $b > 0$ breaks down at $u = 1$. Namely, the radial FG coordinate corresponds to geodesics which do not reach beyond $u = 1$. If one continues to affine parameter values greater than $\tau*$, where $u(\tau*) = 1$, the geodesics bounce back and they cannot be used to define a coordinate system since they doubly cover the region $u < 1$ as is shown in figure 2. Therefore the FG coordinates are well-defined for affine parameter values $\tau < \tau*$. This means that $|x| = e^{-\tau}$ must be bounded below. Another way to see this is to observe that (B.15) implies that x^2 is a monotonically decreasing function of u. Hence the upper bound $u < 1$ on u implies a lower bound on x^2. Setting $u = 1$ in (B.16) and solving for x to leading order in b we find[14] $|x| > x_o = b/\sqrt{2} + \mathcal{O}(b^2)$. Therefore $x = 0$ is *not* part of the manifold and hence the metric (B.5) (and (B.13)) is *non-singular* in the region it is well-defined. (To cover the entire spacetime one would have to use another coordinate patch that covers the deep interior region $1 \leq u \leq u_o$, but this is irrelevant for our holographic computations.) Notice that the bound on x translates into $|z_d| > x_o z_o$ and so only at the boundary $z_o \to 0$ does z_d cover the entire real line. More crucially, the bound $z_o < |z_d|/x_o$ means that the radial coordinate, z_o, is well-defined everywhere *except* at $z_d = 0$, which precisely corresponds to the corner where the two-halves of the boundary meet. This can also be seen directly from the properties of type (i) geodesics. Since $\tilde{z}_o \to |z_d|$ as $\tau \to -\infty$ with z_d constant along the geodesics, we have

$$\tilde{z}_o(\tau) = |z_d| \exp \int_{-\infty}^{\tau} d\tau' u(\tau')\,. \tag{B.29}$$

[14]One must be careful since the hypergeometric function $F(d, 2; d+1; \frac{1}{1+x^2})$ is singular at $x^2 = 0$. See eq. 15.3.12 in [32].

So for $b > 0$ the FG geodesics hit again the boundary $u = 0$ at $\tilde{z}_o = |z_d|\alpha$, where $\alpha = \exp \int_0^1 \frac{du}{\sqrt{(1-u)(1-u+bu^d)}}$. At $z_d = 0$ these geodesics become degenerate and they stay along $\tilde{z}_o = 0$. Since the entire $\tilde{z}_o = 0$ subspace is mapped to $(z_o, z_d) = (0, 0)$ in the FG coordinates, the FG geodesics do not leave the origin once $z_d = 0$. Hence, the FG radial coordinate is not defined at $z_d = 0$.

Finally let us discuss the possibility to use the geodesics of type (ii) in order to define FG coordinates. This is also a natural choice as these geodesics are the obvious generalization of the pure AdS case. Following radial type (ii) geodesics for the Janus geometry, the exact FG metric is

$$ds^2 = \frac{dz_o^2}{z_o^2} + \frac{u_o(u_o - u)}{uz_d^2}dz_d^2 + \frac{dz_a^2}{u\tilde{z}_o^2} \tag{B.30}$$

where

$$\log\left(\frac{z_o^{2/\sqrt{u_o}}}{z_d^2}\right) = \int^u \frac{du}{u\sqrt{(u_o - u)(1 - u + bu^d)}} \tag{B.31}$$

and

$$\tilde{z}_o^2 = z_d^{2u_o} \exp \int_0^u \frac{du}{\sqrt{(u_o - u)(1 - u + bu^d)}}\,. \tag{B.32}$$

Asymptotically, i.e. as $z_o \to 0$,

$$ds^2 \sim \frac{1}{z_o^2}\left[dz_o^2 + (\tilde{z}_d^2)^{\frac{1}{\sqrt{u_0}}-1}(d\tilde{z}_d^2 + dz_a^2)\right] \tag{B.33}$$

where $\tilde{z}_d = z_d^{u_0}$. So when $u_o = 1$, i.e. the pure AdS case, we get the standard result but for the Janus solution these geodesics lead to a non-flat representative of the conformally flat conformal structure. One can perform the additional change of variables given in (8)-(9) of [37] to change the representative to the flat metric. Notice however that the conformal factor is singular at $z_d = 0$ so the corresponding coordinate transformation is singular there. We thus arrive again at the conclusion that the FG coordinates are not well defined at the corner.

We have therefore determined the exact form of the Fefferman-Graham metric for the Janus geometry and we have shown that is well-defined everywhere except on the defect where the two half-boundaries are joined and it is non-singular where it is defined. Moreover, we have shown that the FG metric possesses an inversion isometry which enhances the isometry group to the full $O(1, d)$ isometry group of the original Janus metric. This is reflected in the fact that the holographic calculation gives a zero vev for the stress tensor, which is consistent with a boundary QFT invariant under conformal transformations leaving the plane $z_d = 0$ invariant.

C. Two-point functions for Janus from scalar momentum

Here we give some details of the calculation of the two-point functions for the Janus background based on the scalar equation of motion (4.23). The non-trivial part of the calculation consists in casting the source in a form which significantly reduces the amount of work required. To this end we again use the fact that $\phi_B(x)$ is a function of $x = z_d/z_o$ only and it satisfies

$$\ddot{\phi}_B + d\dot{\phi}_B + e^{-2r}\Box\phi_B = \mathcal{O}(b) \tag{C.1}$$

to write

$$\partial_j\phi_B(x) = \delta_{jd}\frac{1}{z_d}\dot{\phi}_B\,, \qquad \partial_i\partial_j\phi_B = -(d+1)\delta_{id}\delta_{jd}\frac{\dot{\phi}_B}{z_o^2+z_d^2} + \mathcal{O}(b)\,. \tag{C.2}$$

Decomposing S^i_j into irreducible components then (4.23) becomes

$$\begin{aligned} -\Box_g\phi &= -cz_o\partial_i\left[\frac{1}{(1+x^2)^{\frac{d+1}{2}}}e^i_d\right] + \\ &+\frac{cx}{(1+x^2)^{\frac{d+1}{2}}}\Bigg\{-\frac{z_o^2}{2(d-1)}\Box f + \frac{(d+1)}{1+x^2}\left[2\partial_{z_d}\epsilon_d + \frac{d}{d-1}\left(\frac{1}{d}-\frac{\partial^2_{z_d}}{\Box}\right)f + \frac{\partial^2_{z_d}}{\Box}S\right] - \\ &\qquad -\frac{z_o^2}{z_d}\left[\Box\epsilon_d - \partial_{z_d}f + \frac{1}{2}\partial_{z_d}S\right]\Bigg\} + \mathcal{O}(b)\,. \end{aligned} \tag{C.3}$$

Quite remarkably, this can be cast in the form

$$-\Box_g\phi = -cz_o\partial_i\left[\frac{1}{(1+x^2)^{\frac{d+1}{2}}}e^i_d\right] - \Box_g\tilde{J}_\phi + \mathcal{O}(b)\,, \tag{C.4}$$

where

$$\tilde{J}_\phi = \frac{cz_o^d}{(z_o^2+z_d^2)^{\frac{d+1}{2}}}\left[\alpha + \epsilon_d + \frac{1}{2}\frac{\partial_{z_d}}{\Box}S + \frac{1}{2(d-1)}\left(z_d - d\frac{\partial_{z_d}}{\Box}\right)f\right], \tag{C.5}$$

and α is a constant. Hence, the inhomogeneous solution is

$$\phi = \tilde{J}_\phi - c\int d^{d+1}w\sqrt{g(w)}G(\xi)\partial_i\left[\frac{z_o}{(1+x^2)^{\frac{d+1}{2}}}e^i_d\right], \tag{C.6}$$

where e^i_d is given by the zero-order solution

$$e^i_j(z) = \int d^dyK_d(z,\vec{y})e_{(0)}{}^i_j(\vec{y})\,. \tag{C.7}$$

This expression for ϕ immediately gives the canonical momentum from which we obtain the two-point function by differentiating w.r.t. $S_{(0)}{}^i_j$.[15] The result is

$$\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle = -2(d+1)c\Pi^i{}_d{}^d{}_jI(\vec{z},\vec{w}) - \tag{C.8}$$

[15]Note that

$$\begin{aligned} \langle T_{ij}(x)\mathcal{O}(x')\rangle &= \left(-\frac{1}{\sqrt{g_{(0)}(x')}}\frac{\delta}{\delta\phi_{(0)}(x')}\right)\left(-\frac{2}{\sqrt{g_{(0)}(x)}}\frac{\delta}{\delta g_{(0)}{}^{ij}(x)}\right)W \\ &= \left(-\frac{1}{\sqrt{g_{(0)}(x')}}\frac{\delta}{\delta\phi_{(0)}(x')}\right)\langle T_{ij}(x)\rangle \\ &= \left(-\frac{2}{\sqrt{g_{(0)}(x)}}\frac{\delta}{\delta g_{(0)}{}^{ij}(x)}\right)\langle\mathcal{O}(x')\rangle + \delta^{(d)}(x,x')g_{(0)ij}(x)\langle\mathcal{O}(x')\rangle\,, \end{aligned}$$

so if one starts the computation from the scalar 1-point functions, one should remember to include the contact term given above to obtain the full expression.

$$-\frac{cd}{|w_d|^{d+1}}\left[2\pi^{(i}_d\frac{\partial_{j)}}{\Box}+\delta^i_j\frac{\partial_d}{\Box}+\frac{1}{d-1}\left(w_d-d\frac{\partial_d}{\Box}\right)\pi^i_j-\frac{1}{d}w_d\delta^i_j\right]_w\delta^{(d)}(\vec{z}-\vec{w}).$$

It is a straightforward exercise to verify that this is equivalent to the two-point functions given above, as calculated from the graviton momentum.

D. Short distance expansion of the holographic two-point function $\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle$

For the convenience of the reader we will give here the essential steps required to evaluate the short distance expansion of the transverse traceless part of the holographic two-point function $\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle$, namely $\Pi^i{}_d{}^d{}_j I(\vec{z},\vec{w})$, where $I(\vec{z},\vec{w})$ is given in (4.36).[16]

First, after a shift and rescaling of the integration variables, $I(\vec{z},\vec{w})$ can be written as

$$\begin{aligned}I(\vec{z},\vec{w}) &= \frac{c_d^2}{(\vec{s}^2)^{d/2-1}}\int_0^\infty dx_o x_o^{2d+1}\int d^dx\frac{w_d+|\vec{s}|x_d}{[x_o^2\vec{s}^2+(w_d+|\vec{s}|x_d)^2]^{\frac{d+3}{2}}}\times\\ &\times\frac{1}{[x_o^2+\vec{x}^2]^d}\frac{1}{[x_o^2+(\vec{x}-\hat{s})^2]^d},\end{aligned} \tag{D.1}$$

where $\hat{s}=\vec{s}/\vec{s}^2$. This form is suitable for a short distance expansion in $|\vec{s}|$. Each term in the expansion can be explicitly evaluated using the standard Feynman parameters technique. The result to leading order is given in (4.45).

To evaluate the projection operator on this expression we use the fact that

$$\frac{1}{(\vec{s}^2)^\alpha}=-\frac{1}{2(\alpha-1)(d-2\alpha)}\Box\frac{1}{(\vec{s}^2)^{\alpha-1}}, \tag{D.2}$$

for any power $\alpha\neq d/2$ in order to cancel the $1/\Box$ factors in the projection operator. It is then straightforward to evaluate the derivatives in the numerator of the projection operator to obtain the short distance expansion of the transverse traceless part. The result is given in section 4.4.

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